The Universal Lie algebra

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Abstract. The Kontsevich integral of a knot K lies in an algebra of diagrams $\mathcal{A}_c(S^1)$. This algebra is (up to completion) a symmetric algebra of a graded module $\mathcal{P}$, where $\mathcal{P}$ is the set of primitive elements of $\mathcal{A}_c(S^1)$. The elements of $\mathcal{P}$ are represented by S^1-diagrams K such that the complement of the circle in K is connected and non empty. On the other hand there is an isomorphism from $\oplus\mathcal{B}_n$ to $\mathcal{A}_c(S^1)$, where $\mathcal{B}_n$ is the module generated by uni-trivalent diagrams whith n uni-valent vertices, and divided by the AS and IHX-relations. Actually this isomorphism induces an isomorphism from the direct sum of modules $\mathcal{B}'_n$, $n > 0$ to $\mathcal{P}$, where $\mathcal{B}'_n$ is the submodule of $\mathcal{B}_n$ generated by connected diagrams. Therefore to understand $\mathcal{A}_c(S^1)$, it's enough to describe the modules $\mathcal{B}'_n$.

These modules are part of a more complicated object: the category of diagrams $\mathcal{D}$. This category is a monoidal linear category. Every Lie algebra L equipped with a non singular symmetric invariant bilinear form induces a functor from $\mathcal{D}$ to the category of L-modules and, roughly speaking, these functors are the only one known.

The purpose of this paper is to construct a monoidal category which looks like the category of module over a Lie algebra and which is universal in some sense. A lot of properties of this category is shown and many conjectures are given. In some sense this category is the universal Lie algebra, and every simple Lie (super)algebra is obtained by changing the coefficient ring.

1. THE ALGEBRA Λ

1.1 The construction of Λ

Let Γ be a curve (i.e. a one-dimensional compact manifold), and X be a finite set. Denote by $\mathcal{A}(\Gamma, X)$ the **Q**-module generated by all (Γ, X)-diagrams and divided by the AS and IHX-relations (see [V1] for definitions). If Γ is empty, the module

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$\mathcal{A}(\Gamma, X)$ will be simply denoted by $\mathcal{D}(X)$. Denote also by $\mathcal{D}_c(X)$ the submodule of $\mathcal{D}(X)$ generated by connected $(\emptyset, X)$-diagrams, and by $\mathcal{D}_s(X)$ the submodule of $\mathcal{D}(X)$ generated by connected non-empty $(\emptyset, X)$-diagrams having at least one 3-valent vertex. It is easy to see the following:

1.2 Proposition. *Let X be a finite set. Let $\pi(X)$ be the set of partitions of X. Then there is a canonical isomorphism:*

$$\mathcal{D}(X) = \mathcal{D}(\emptyset) \otimes \Big(\bigoplus_{\pi \in \pi(X)} \ \bigotimes_{Y \in \pi} \mathcal{D}_c(Y) \Big)$$

If X has 0 or 2 elements, one has:

$$\mathcal{D}_c(X) \simeq \mathbf{Q} \oplus \mathcal{D}_s(X)$$

If X has one element, $\mathcal{D}_c(X)$ and $\mathcal{D}_s(X)$ are trivial modules. If X has at least 3 elements the two modules $\mathcal{D}_c(X)$ and $\mathcal{D}_s(X)$ are equals.

Proof: The first formula is a consequence of the fact that a $(\emptyset, X)$-diagram K may be written in a unique way as a disjoint union: $K = H \cup \big(\cup_i K_i\big)$, where H has non univalent vertex, and K_i are connected and non-empty. The sets $X \cap K_i$ form a partition of X, and the formula follows.

On the other hand every non-empty connected $(\emptyset, X)$-diagram has a 3-valent vertex except the circle if X is empty or the interval $[0, 1]$ if X has 2 elements. The fact that $\mathcal{D}_c(X) = \mathcal{D}_s(X) = 0$ when X has only one element, is an easy exercice (see [V2] for a proof). □

If X is a set, the symmetric group $\mathfrak{S}(X)$ acts on modules $\mathcal{D}(X)$, $\mathcal{D}_c(X)$ and $\mathcal{D}_s(X)$. In particular for every $n > 0$ the module $F(n) = \mathcal{D}_s([n])$ is a $\mathfrak{S}_n$-module.

1.3 Definition. Let Λ be the submodule of $F(3) = \mathcal{D}_s([3])$ consisting of all elements $u \in F(3)$ satisfying the following:

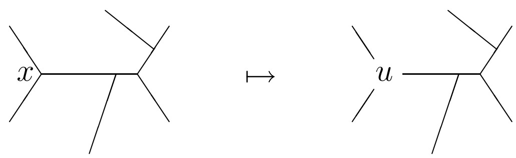
$$\forall \sigma \in \mathfrak{S}_3, \qquad \sigma(u) = \varepsilon(\sigma) u$$

where ε is the signature homomorphism. The degree of an element $u \in \Lambda$ represented by a diagram K is $(n-4)/2$, where n is the number of vertices of K. This degree is also the rank of $\mathrm{H}_1(K)$. With this degree, Λ is a graded $\mathbf{Q}$-module.

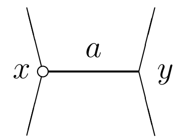
1.4 Proposition. *The module Λ is actually a graded $\mathbf{Q}$-algebra. Moreover, for every set X, $\mathcal{D}_s(X)$ is equipped with a natural Λ-algebra structure.*

Proof: Let X be a finite set. Let K be a $(\emptyset, X)$-diagram. Suppose that K is connected and has some 3-valent vertex x. Let u be an element of Λ represented by a $(\emptyset, [3])$-diagram H. Because of the numbering of the set of edges arriving to x, one can insert H in K near x and one gets a new diagram $K(x, H)$. Since H is completely antisymmetric with respect with the $\mathfrak{S}_3$-action, the class of $K(x, H)$ in

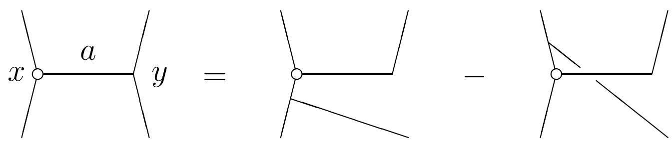
$\mathcal{D}(X)$ doesn't depend on the choice of the numbering. Moreover it depends only on K, x and u and will be denoted by $K(x, u)$.



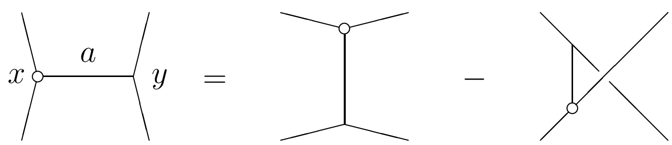
Consider an edge a in K with vertices x and y. Consider the following part of $K(x, u)$, where the small circle represents H inserted near x:



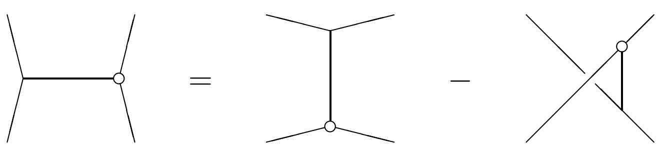
Because of the next lemma the bottom right edge may cross H and we have in $\mathcal{D}_s(X)$:



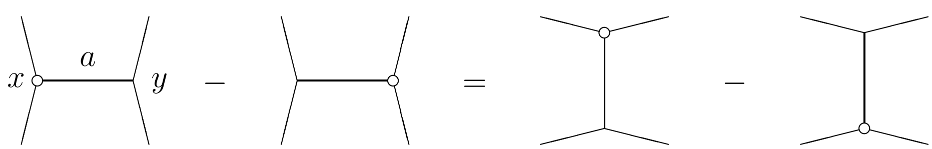
Or equivalently:



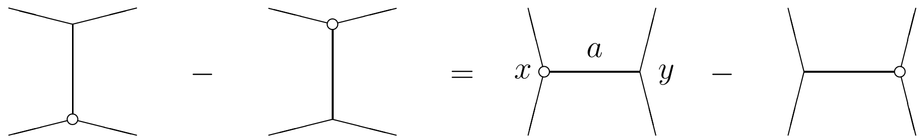
In the same way we have:



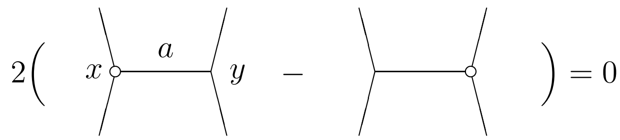
which implies:



and, by applying a rotation of the picture, we have:



and then:

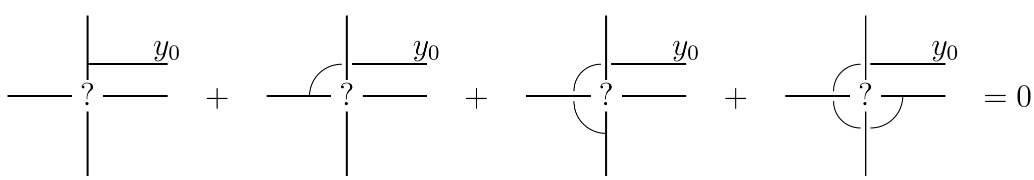


Therefore inserting H near x or y gives the same element in $\mathcal{D}_s(X)$ and the element $K(x,u)$ doesn't depend on the choice of the vertex x. Then $K(x,u)$ depends only on K and the class u of H in Λ. It is easy to see that the map $K \mapsto K(x,u)$ is compatible with the AS relation. But this transformation is also compatible with the IHX relation because such a relation corresponds to an edge a in K and the transformation may be done near a vertex outside a. If K has only two vertices this proof doesn't work but a direct computation shows also the compatibility with the IHX relation.

Hence this transformation induces a well defined homomorphism from $\Lambda \otimes \mathcal{D}_s(X)$ to $\mathcal{D}_s(X)$. In particular this homomorphism induces a morphism from $\Lambda \otimes \Lambda$ to Λ and Λ becomes an algebra. It is easy to see that the previous morphism from $\Lambda \otimes \mathcal{D}_s(X)$ to $\mathcal{D}_s(X)$ induces on $\mathcal{D}_s(X)$ a structure of Λ-module. So the last thing to do is to prove the following lemma:

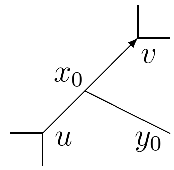
1.5 Lemma. *Let X be a finite set and Y be the set X with one extra point y_0 added. Let K be a connected $(\emptyset, X)$-diagram. For every $x \in X$ denote by K_x the $(\emptyset, Y)$-diagram obtained by adding to K an extra edge from y_0 to a point in K near x, the cyclic ordering near the new vertex being given by taking the edge coming from y_0 first, the edge coming from x after and the last edge at the end.*

Then the element $\sum_x K_x$ is trivial in the module $F(Y)$.



Proof: For every oriented edge a of K from a vertex u to a vertex v, we can connect y_0 to K by adding an extra edge from y_0 to a new vertex x_0 in a and we get a $(\emptyset, Y)$-diagram K_a where the cyclic order between edges arriving at x_0 is

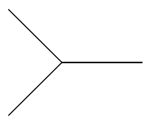
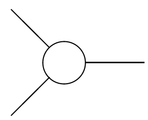
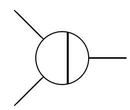
(x_0u, x_0y_0, x_0v).

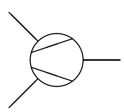
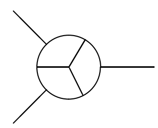


It is clear that the expression $K_a + K_b$ is trivial if b is the edge a with the opposite orientation. Moreover if a, b and c are the three edges starting from a 3-valent vertex of K, the sum $K_a + K_b + K_c$ is also trivial. Therefore the sum $\sum K_a$ for all oriented edge a of K is trivial and is equal to the sum $\sum K_a$ for all oriented edge a starting from a vertex in X. That proves the lemma. □

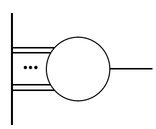
Remark. The algebra is commutative. In [V2] the algebra Λ is constructed with integral coefficients and it is shown that $12ab = 12ba$ for every a and b in Λ. In this situation Λ is defined over the rationals and is commutative.

In degree less to 4, the module Λ is generated (over $\mathbf{Q}$) by the following diagrams:

$1 =$  $t =$  $t^2 =$ 

$t^3 =$  $x_3 =$ 

Let $n > 0$ be an integer. We have the following element in $F(3)$:

$x_n =$ 

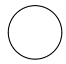
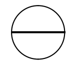
having n horizontal edges on the left hand side of the picture. It is proven in [V2] that x_n lies in Λ for every $n > 0$. We have:

$$x_1 = 2t \qquad \text{and} \qquad x_2 = t^2$$

Moreover the even $x_n's$ can be express in term of the odd $x_n's$.

1.6 Proposition. *The algebra $End_{\mathcal{D}}([0])$ of endomorphisms of the emptyset $[0]$ in the category $\mathcal{D}$ is isomorphic to the tensor product of the polynomial algebra $\mathbf{Q}[\delta]$ and the symmetric algebra $S(E)$ of the free Λ-module E generated by the Θ-diagram.*

Proof: It is clear that $\mathrm{End}_{\mathcal{D}}([0])$ is the symmetric algebra of the module $\mathcal{D}_c([0])$ of connected non-empty diagrams. The module $\mathcal{D}_c([0])$ is the direct sum of the $\mathbf{Q}$-module generated by the circle δ and the module $\mathcal{D}_s([0])$. But this last module is equipped with a Λ-module structure. Actually $\mathcal{D}_s([0])$ is the free Λ-module generated by the Θ-diagram with two vertices and three edges joining them. The result follows. □

$\delta =$  $\Theta =$ 

Only partial things are known about the structure of Λ. Every simple quadratic Lie (super)algebra L produces an algebra homomorphism from Λ to the coefficient ring of L (see next section). By this way one gets 8 algebra homomorphisms from Λ to different polynomial algebras. Another point which is known is the following: the elements x_1, x_3, x_5, ... are not algebraically independant. A family of relations including a special relation in degree 10 considered in [V2] was discovered by Kneissler [K]. In order to explain these relations, one has to consider the following algebras:

Let α, β and γ be formal variables of degree 1. Let R be the algebra of symmetric polynomials in α, β and γ. This algebra R is a subalgebra of $\mathbf{Q}[\alpha,\beta,\gamma]$. If $t=\alpha+\beta+\gamma$, $s=\alpha\beta+\beta\gamma+\gamma\alpha$ and $p=\alpha\beta\gamma$ are the elementary symmetric polynomials, R is the algebra $\mathbf{Q}[t,s,p]$. Set: $\omega=(t+\alpha)(t+\beta)(t+\gamma)=p+st+2t^3$ and define R_0 to be the subalgebra $\mathbf{Q}[t]\oplus\omega R$ of R.

On the other hand consider the elements $x'_n\in R$, $n\geq 0$ defined by the following:

— $x'_0=0$ $\quad x'_1=2t$ $\quad x'_2=t^2$

— $\forall n\geq 0$ $\quad x'_{n+3}=tx'_{n+2}-sx'_{n+1}+px'_n+\dfrac{st^{n+1}}{2}-\dfrac{pt^n}{2}-p(2t)^n$

It is an easy exercice to check that all these elements belong to the subalgebra R_0.

With these algebras the result of Kneissler may be express in the following way:

1.7 Theorem. *There exists a unique homomorphism φ of graded algebras from R_0 to Λ satisfying the following conditions:*

— *φ sends t to t*

— *for every $n>0$, φ sends x'_n to x_n.*

Related with this result we can formulate different conjectures:

1.8 Conjecture. *The morphism φ is injective.*

1.9 Conjecture. *The morphism φ is bijective.*

Presently φ is known to be bijective in degree <11 and injective in degree <16.

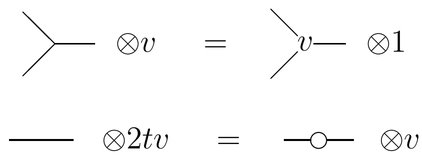
2. THE CATEGORY $\mathcal{D}'$

The algebra Λ acts on many modules of diagrams but not on all. In particular the set of morphisms $\mathrm{Hom}_{\mathcal{D}}([p],[q])$ in $\mathcal{D}$ between two object $[p]$ and $[q]$ are not Λ-modules.

The category $\mathcal{D}'$ is the category $\mathcal{D}$ where an action of Λ is forced. More precisely the objects of the category $\mathcal{D}'$ are the sets $[p]$, $n \geq 0$ and if $[p]$ and $[q]$ are two objects in $\mathcal{D}$ (or $\mathcal{D}'$), the module $\mathrm{Hom}_{\mathcal{D}'}([p],[q])$ of morphisms in $\mathcal{D}'$ from $[p]$ to $[q]$ is defined to be the quotient of $\mathrm{Hom}_{\mathcal{D}}([p],[q]) \otimes \Lambda$ by the following relations:

— Let u is an element of $\mathrm{Hom}_{\mathcal{D}}([p],[q])$ represented by a diagram K, and x be a 3-valent vertex in K. Let v be an element in Λ. Then $u \otimes v$ is equivalent to $u' \otimes 1$ where u' is obtained from u by inserting v near x.

— Let u is an element of $\mathrm{Hom}_{\mathcal{D}}([p],[q])$ represented by a non empty diagram K and v be an element in Λ. Then $u \otimes 2tv$ is equivalent to $u' \otimes v$ where u' is obtained from u by inserting a circle in some edge in K.



2.1 Proposition. *The category $\mathcal{D}'$ is a linear monoidal category over the polynomial algebra $\Lambda[\delta]$. Moreover the canonical functor from $\mathcal{D}$ to $\mathcal{D}'$ induces, for every $[p]$ and $[q]$ a morphism from $\mathrm{Hom}_{\mathcal{D}}([p],[q])$ to $\mathrm{Hom}_{\mathcal{D}'}([p],[q])$ which is injective on the submodule of $\mathrm{Hom}_{\mathcal{D}}([p],[q])$ generated by connected diagrams.*

Proof: If we apply the construction above to the functors $\mathcal{D}$, $\mathcal{D}_c$ and $\mathcal{D}_s$ considered in section 1.1, we get new functors $\mathcal{D}'$, $\mathcal{D}'_c$ and $\mathcal{D}'_s$ from finite sets to Λ-modules. By construction, $\mathcal{D}'_s(X)$ is isomorphic to $\mathcal{D}_s(X)$ for every finite set X. If X is finite, one has: $\mathcal{D}_c(X) = \mathbf{Q}^n \oplus \mathcal{D}_s(X)$, where $n = 1$ if the order of X is 0 or 2 and $n = 0$ otherwise. Therefore we get:

$$\mathcal{D}'_c(X) = \Lambda^n \oplus \mathcal{D}_s(X)$$

and $\mathcal{D}_c(X)$ is contained in $\mathcal{D}'_c(X)$.

The module $\mathcal{D}(\emptyset)$ is actually an algebra by the disjoint union of diagrams. More precisely $\mathcal{D}(\emptyset)$ is the symmetric algebra of the graded module $\mathcal{D}_c(\emptyset)$:

$$\mathcal{D}(\emptyset) = S(\mathcal{D}_c(\emptyset)) = \mathbf{Q}[\delta] \otimes S(\mathcal{D}_s(\emptyset)) = \mathbf{Q}[\delta] \otimes S(\Lambda\Theta) = \mathbf{Q}[\delta] \otimes S(2t\Lambda\delta)$$

The module $\mathcal{D}'(\emptyset)$ is also an algebra, but over Λ:

$$\mathcal{D}'(\emptyset) = \Lambda[\delta]$$

For an arbitrary finite set X, we have:

$$\mathcal{D}(X) = \mathcal{D}(\emptyset) \otimes \Big(\bigoplus_{\pi \in \pi(X)} \bigotimes_{Y \in \pi} \mathcal{D}_c(Y) \Big)$$

$$\mathcal{D}'(X) = \Lambda[\delta] \otimes_\Lambda \Big(\bigoplus_{\pi \in \pi(X)} \bigotimes_{Y \in \pi} \mathcal{D}'_c(Y) \Big)$$

where the last tensor product is over Λ.

By applying this results to the morphisms of the categories $\mathcal{D}$ and $\mathcal{D}'$, one gets the result. The action of Λ on modules of homomorphisms is obtained by construction. The multiplication by δ is the disjoint union with a circle.

2.2 Proposition. *Let L be a quadratic Lie (super)algebra over a coefficient field k. Suppose L is simple with a non trivial bracket. Then there exists a unique algebra homomorphism χ_L from $\Lambda[\delta]$ to k and a unique functor Φ_L of monoidal categories from $\mathcal{D}'$ to the category Mod_L of L-modules such that:*

— *$\chi_L(\delta)$ is the (super)dimension of L*

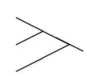
— *Φ_L sends [1] to the adjoint representation L and the following diagrams:*

to the scalar product, the Casimir element, the Lie bracket and the (super)symmetry respectively

— *For every morphism $f \in \mathcal{D}'$ and every $v \in \Lambda[\delta]$ one has:*

$$\Phi_L(vf) = \chi_L(v)\Phi_L(f)$$

Proof: In [V1] a functor Φ from the category $\mathcal{D}$ to Mod$\mathcal{L}$ is constructed. It satisfies all the properties above except the properties relative to Λ and δ. Let v be an element in Λ considered as a morphism in $\mathcal{D}$ from [2] to 1. Denote by φ the homomorphism from $L \otimes L$ to L induced under Φ by v. Consider the following morphism D from [3] to [1]:



The image of D is the morphism:

$$x \otimes y \otimes z \in L^{\otimes 3} \mapsto [[x, y], z]$$

If we multiply D by v in the two different ways we obtain, for every x, y and z in L:

$$\varphi([x, y], z) = [\varphi(x, y), z]$$

Therefore $\varphi(x, y)$ depends only on the bracket $[x, y]$ and there exists an endomorphism f of L such that: $\varphi(x, y) = f([x, y])$. Since L is supposed to be simple, f is the multiplication by an element $a = \chi_L(v) \in k$ which depends only on v. It is easy to check that χ_L is a homomorphism of algebras and satisfies the following:

$$\Phi(vD) = \chi_L(v)\Phi(D)$$

for every $v \in \Lambda$ and every diagram D for which vD is defined.

On the other hand Φ transform the circle considered as a morphism from $[0]$ to itself to the multiplication by the dimension d of L. Therefore the functor Φ factorizes through the category $\mathcal{D}'$ by a functor Φ_L satisfying all the desired properties. $\square$

The algebra homomorphism χ_L is described in [V2] for every simple quadratic Lie (super)algebra. We get the following:

— If L is the Lie superalgebra $sl(E)$ where E is a super k-module of superdimension n, the character χ_L restricted to R_0 is obtained by sending α, β and γ to n, 2 and -2.

— If L is the Lie superalgebra $osp(E)$ where E is a super k-module of superdimension n equipped with a non singular supersymmetric bilinear form, the character χ_L restricted to R_0 is obtained by sending α, β and γ to $n-4$, 4 and -2.

— If L is a Lie superalgebra of type D(2, 1,?), the character χ_L restricted to R_0 is obtained by sending α, β and γ to arbitrary elements in the coefficient field with the only condition: $\alpha+\beta+\gamma=0$.

— If L is an exceptional Lie algebra of type E6, E7, E8, F4 or G2, the character χ_L restricted to R_0 is obtained by sending (α,β,γ) to $(3,-1,4)$, $(4,-1,6)$, $(6,-1,10)$, $(5,-2,6)$ and $(5,-3,4)$ respectively.

There are few other examples of quadratic simple Lie superalgebras. The character corresponding to $psl(n,n)$ may be defined in term of sl characters. Lie superalgebras G(3) and F(4) induce the same character as sl_2 and sl_3. The Hamiltonian algebras induce the augmentation character.

The characters χ_L where L is of sl type fit together in one graded algebra homomorphism χ_{sl} from Λ to $R/(sl)$ where (sl) is the ideal of R generated by the polynomial $P_{sl}=\prod(\alpha+\beta)=p-st$. The osp-type characters fit together in one graded algebra homomorphism χ_{osp} from Λ to $R/(osp)$ where (osp) is the ideal of R generated by the polynomial $P_{osp}=\prod(\alpha+2\beta)=8s^2t^2+4s^3+4pt^3-18pst+27p^2$. In the same way the $D(2,1,\alpha)$-type characters induce a graded algebra homomorphism χ_{sup} from Λ to $R/(sup)$ where (sup) is the ideal of R generated by the polynomial $P_{sup}=t$. The exceptional Lie algebras induce graded algebra homomorphisms χ_i from Λ to $R/(exc_i)$ where (exc_i) is the ideal of R generated by $P_{exc}=\prod(3\alpha-2t)=4t^3-18st+27p$ and the polynomial P_i equal to $36s-5t^2$, $81s-14t^2$, $225s-44t^2$, $81s-8t^2$ or $36s+7t^2$ if the Lie algebra is E6, E7, E8, F4 or G2. A last interesting character is obtained by the Lie algebra sl_2. It can be seen as a graded algebra homomorphism from Λ to $R/(sl_2)$ where (sl_2) is the ideal generated by the polynomial $P_{sl_2}=\prod(t+\alpha)=\omega=p+st+2t^3$.

All these characters are compatible in the following sense:

2.3 Theorem [P]. *Let I be the intersection in R of the ideals (sl), (osp), (sup), (exc_i) and (sl_2). Then all the character above induce a graded algebra homomorphism χ from Λ to R_0/I. Moreover the composite $\chi\circ\varphi$ from R_0 to R_0/I is the quotient homomorphism.*

Remark. Since the first element in I is the product $P_{sl}P_{osp}P_{sup}P_{exc}P_{sl_2}$ which is a polynomial of degree 16, the first element in R_0 which may be killed in Λ is this polynomial in degree 16.

3. THE UNIVERSAL LIE ALGEBRA

Pseudo quadratic Lie algebra.

Let L be a quadratic Lie (super)algebra over a commutative ring k. Let Mod_L be the category of L-modules. This category is monoidal and k-linear. The adjoint representation still denoted by L is a particular module in this category. On the other hand the scalar product $f_1 = <?,?>$, the Casimir element $f_2 = \Omega$, the Lie bracket $f_3 = [?,?]$ and the (super)symmetry $f_4 = T$ are homomorphisms in Mod_L from $L^{\otimes 2}$ to $L^{\otimes 0}$, from $L^{\otimes 0}$ to $L^{\otimes 2}$, from $L^{\otimes 2}$ to $L^{\otimes 1}$ and from $L^{\otimes 2}$ to $L^{\otimes 2}$ respectively.

Moreover we have the following properties:

— $f_3 \circ f_4 = -f_3$

— $f_3 \circ (f_3 \otimes 1) \circ \Big(1 \otimes 1 \otimes 1 + (f_4 \otimes 1) \circ (1 \otimes f_4) + (1 \otimes f_4) \circ (f_4 \otimes 1)\Big) = 0$

— $f_1 \circ f_4 = f_1$

— $f_1 \circ (f_3 \otimes 1) = f_1 \circ (1 \otimes f_3)$

— $f_4 \circ f_4 = 1 \otimes 1$

— $1 = (f_1 \otimes 1) \circ (1 \otimes f_2) = (1 \otimes f_1) \circ (f_2 \otimes 1)$

— $(f_4 \otimes 1) \circ (1 \otimes f_4) \circ (f_4 \otimes 1) = (1 \otimes f_4) \circ (f_4 \otimes 1) \circ (1 \otimes f_4)$

— $(1 \otimes f_3) \circ (f_4 \otimes 1) \circ (1 \otimes f_4) = f_4 \circ (f_3 \otimes 1)$

— $(1 \otimes f_1) \circ (f_4 \otimes 1) = (f_1 \otimes 1) \circ (1 \otimes f_4)$

The category Mod_L is not strictly associative. But the full subcategory Mod'_L of Mod_L generated by the tensor products of L contains the morphisms f_i and is strictly associative.

Definition. Let k be a commutative ring. A pseudo quadratic Lie algebra L over k is a monoidal k-linear category $\mathcal{L}$ equipped with an object L and four morphisms f_1, f_2, f_3 and f_4 such that:

— the objects of $\mathcal{L}$ are the objects $L^{\otimes n}$, $n \geq 0$

— f_1 is a morphism from $L^{\otimes 2}$ to $L^{\otimes 0}$

— f_2 is a morphism from $L^{\otimes 0}$ to $L^{\otimes 2}$

— f_3 is a morphism from $L^{\otimes 2}$ to $L^{\otimes 1}$

— f_4 is a morphism from $L^{\otimes 2}$ to $L^{\otimes 2}$

— the morphisms f_i satisfy the nine properties above.

For simplicity the unit object $L^{\otimes 0}$ will be also denoted by k.

Definition. Let k and k' be commutative rings. Let $L = (L, f_1, f_2, f_3, f_4)$ and $L' = (L', f'_1, f'_2, f'_3, f'_4)$ be two pseudo quadratic Lie algebras over k and k'. A morphism from L to L' is a ring homomorphism χ from k to k' together with a functor of monoidal categories Φ from L to L' sending L to L' and morphisms f_i to f'_i and such that Φ is linear over χ on the modules of homomorphisms.

Remarks. Let L be a quadratic Lie (super)algebra. Then the category Mod'_L satisfy all the properties of a pseudo quadratic Lie algebra. In this sense, a quadratic Lie (super)algebra is a particular pseudo quadratic Lie algebra.

Because of this canonical example the morphism f_1 is called the scalar product, f_2 the casimir element, f_3 the Lie bracket and t_4 the symmetry.

The categories of diagrams $\mathcal{D}$ and $\mathcal{D}'$ are particular examples of pseudo quadratic Lie algebras. The first one is over $\mathbf{Q}$ and the second one over $\Lambda[\delta]$.

3.1 Theorem. *Let L be a pseudo quadratic Lie algebra over a $\mathbf{Q}$-algebra k. Then there exists a unique morphism Φ from $\mathcal{D}$ to L.*

Sketch of proof: The functor is obviously defined on the objects. On the coefficients ring it's the unique ring homomorphism from $\mathbf{Q}$ to k. To define Φ on the modules of morphisms, it is enough to defined $\Phi(D)$ where D is a diagram which represents a morphism from an object $[p]$ to another object $[q]$. Consider $[p]$ included in the standard way in $\mathbf{R}\times\{0\}$ and $[q]$ in $\mathbf{R}\times\{1\}$. Let f be a PL map from D to $\mathbf{R}\times[0,1]$ which extends the previous inclusions. If f is chosen to be generic enough, its image doesn't contain any vertical segment and has only finitely many double points. We may also suppose that two vertices or double points are not in a common vertical line. Then, by cutting $f(D)$ by vertical lines, one obtains a decomposition of D as a composite of morphisms of the form $\mathrm{Id}\otimes d_i\otimes\mathrm{Id}$. By using the same expression but with f_i instead of d_i one gets an morphism $\Phi(D)$ from $L^{\otimes p}$ to $L^{\otimes p}$.

Suppose now that g is another generic PL map from D to $\mathbf{R}\times[0,1]$ which satisfies the same condition as above. Then one construct a homotopy h_t between f and g which as generic as possible. For such a homotopy, h_t is generic except for finitely many values of t. For a generic t the corresponding morphism $\Phi(D)_t$ is defined. This function is locally constant and has maybe some jump on non generic t. The non generic values of t correspond to the case where some edge becomes vertical, or a double point (or a vertex) crosses some edge, or two double points (or vertices) have a commun first coordinate. One have to check all these cases, but each of these corresponds to some formula satisfied by the f_i's and the function $t\mapsto\Phi(D)_t$ has no jump. That implies that $\Phi(D)$ doesn't depend on the choice of f. The fact that Φ is compatible with AS and IHX relations is easy to check.

So the functor is defined and the theorem is proven. $\square$

Definition. Let L be a pseudo quadratic Lie algebra over a commutative ring k. Then L is called reduced if the algebra of endomorphisms of $L^{\otimes 0}$ is the module $k\mathrm{Id}$. It is called simple if $\mathrm{End}(L)$ is also the module $k\mathrm{Id}$.

3.2 Theorem. *Let L be a simple pseudo quadratic Lie algebra over a $\mathbf{Q}$-algebra k.*

Suppose that the following diagram is cartesian:

$$\begin{array}{ccc} Hom(L,L) & \longrightarrow & Hom(L,L^{\otimes 2}) \\ \downarrow & & \downarrow \\ Hom(L^{\otimes 2},L) & \longrightarrow & Hom(L^{\otimes 2},L^{\otimes 2}) \end{array}$$

where the horizontal morphisms are the composition from the left with the cobracket (the dual of the bracket), and the vertical morphisms are the composition from the right with the bracket.

Then there exists a unique morphism Φ from $\mathcal{D}'$ to L.

Remark. Actually this condition is allways satisfied if L is a simple quadratic Lie (super)algebra over a field with a non zero bracket.

Proof: Because of the last theorem, there is a unique functor Φ_0 from $\mathcal{D}$ to L which a morphism of pseudo quadratic Lie algebra. We have to prove that Φ_0 factorizes uniquely through the category $\mathcal{D}'$. Let d_3 be the bracket in the category $\mathcal{D}$ and d'_3 be the cobracket. These morphisms are represented by the following diagrams:

Let v be an element in Λ. The morphisms $\Phi_0(vd_3)$ and $\Phi_0(vd'_3)$ ly in $\mathrm{Hom}(L^{\otimes 2}, L)$ and $\mathrm{Hom}(L, L^{\otimes 2})$ respectively. Moreover they induce the same morphism from $L^{\otimes 2}$ to itself. Because of the property of L, there exists a unique morphism f from L to L inducing $\Phi_0(vd_3)$ and $\Phi_0(vd'_3)$. On the other hand, L is supposed to be simple and there exists a unique element $a \in k$ such that f is the morphism $a\mathrm{Id}$. This element a depends only on v. Denote it by $\chi(v)$. It is easy te see that χ is actually an algebra homomorphism from Λ to k.

On the other hand the circle δ induces under Φ_0 the scalar form applied to the Casimir element. This endomorphism of $L^{\otimes 0}$ is the multiplication by an element $d \in k$. So we have a well defined algebra homomorphism from $\Lambda[\delta]$ to k. This homomorphism, still denoted by χ, is the previous χ on Λ and send δ to d.

Now it is easy to see that the functor Φ_0 factorizes in a unique way through $\mathcal{D}'$ and the functor Φ is constructed. $\square$

3.3 Direct summand and dimension.

Let L be a pseudo quadratic Lie algebra. Suppose L is reduced (i.e every endomorphism of the unit object $L^{\otimes 0}$ is scalar). It is possible to construct forms b_X and Casimir elements Ω_X for every object X in L. If $X = L^{\otimes n}$ is an object in L, denote by X^* the object $L^{\otimes n}$ where the componants are written in the opposite order. So we have: $(X \otimes Y)^* = Y^* \otimes X^*$ for every objects X et Y in L. The form b_X is a morphism from $X^* \otimes X$ to $L^{\otimes 0} = k$ and Ω_X is an morphism from k to $X \otimes X^*$.

If X is the object L itself b_X is the scalar form and Ω_X the Casimir element. For general objects we construt b_X and Ω_X by induction:

$$b_{X\otimes Y} = b_X \circ (\mathrm{Id}_X \otimes b_Y \otimes \mathrm{Id}_X)$$

$$\Omega_{X\otimes Y} = (\mathrm{Id}_X \otimes \Omega_Y \otimes \mathrm{Id}_X) \circ \Omega_X$$

The form b_X is a morphism from $X \otimes X$ to $L^{\otimes 0}$ and Ω_X is a morphism from $L^{\otimes 0}$ to $X \otimes X$.

If f is an endomorphism from an object X to itself, one defines its trace by:

$$\tau(f) = b_X \circ (f \otimes \mathrm{Id}_X) \circ \Omega_X$$

This morphism is an endomorphism of the unit object. Since L is supposed to be reduced, this morphism is represented by a number. So the trace $\tau(f)$ of an endomorphism f is an element of the coefficient ring k.

It's an easy exercice to show that τ has the formal properties of a trace. More precisely we have:

3.4 Proposition. *The trace homomorphisms are linear. If f is a morphism from an object X to an object Y and f is a morphism from Y to X, on has: $\tau(f \circ g) = \tau(g \circ f)$.*

If f is an endomorphism of an object X and g is an endomorphism of an object Y, on has: $\tau(f \otimes g) = \tau(f)\tau(g)$.

Let π be a projector, that is an endomorphism of an object X such that: $\pi \circ \pi = \pi$. It is possible to consider π as a projection onto a direct summand X_π. This new object lies in a new category. Formally X_π is the projector π itself and, if π and π' are two projectors in $\mathrm{End}(X)$ and $\mathrm{End}(Y)$ respectively, the set of morphisms $\mathrm{Hom}(X_\pi, X_{\pi'})$ is defined by: $\pi'\mathrm{Hom}(X, X')\pi$. So we have a bigger category which is still a monoidal linear category. In this new category the object X decomposes into a direct sum of two objects: the object X_π and $X_{1-\pi}$. The dimension of the object X_π is simply the trace of the projector π.

In order to simplify the terminology, these new objects are called modules, or L-modules. So every L-module in this category has a dimension. This dimension is an element of the coefficient ring k.

Definition. Let L be a pseudo quadratic Lie algebra over an integral domain k. Let M be a L-module. One said that M is simple (resp. absolutely simple) if $\mathrm{End}(M)$ is a commutative integral domain containing $k\,\mathrm{Id}$ (resp. is contained in a localization of $k\,\mathrm{Id}$)

Examples. If L is a quadratic Lie superalgebra, the trace is the supertrace: the trace of the even part minus the trace of the odd part. The dimension of a module is the superdimension: the dimension of the even component minus the dimension of the odd component.

In the category $\mathcal{D}'$, the dimension of $L = [1]$ is simply the element $\delta \in \Lambda[\delta]$.

3.5 Theorem. *Let $\Lambda[\delta] \to A$ be an algebra homomorphism sending t, ω and $p\omega$ to invertible elements in A. Let $L = [1]$ be the generator module in the category $\mathcal{D}'_A = \mathcal{D}' \otimes A$. Let $\bigwedge^2 L$ be the second exterior power of L. Then $\bigwedge^2 L$ decomposes into a direct sum of three modules X_1, X_2 and E. Moreover the dimensions of these modules are:*

$$dim X_1 = \delta$$

$$dim X_2 = -\frac{\omega^2}{p\omega}\delta$$

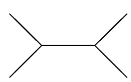
$$dim E = \delta\Big(\frac{\delta-3}{2} + \frac{\omega^2}{p\omega}\Big)$$

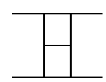
Proof: The elements t, $\omega = p+st+2t^3$ and $p\omega$ are elements of the algebra $R_0 = \mathbf{Q}[t] \oplus \omega R \subset R = \mathbf{Q}[t,s,p]$ and R_0 is sending into Λ by a canonical algebra homomorphism (see 1.7). Then the elements t, ω and $p\omega$ belong to Λ and becomme invertible in A. The category $\mathcal{D}'_A$ is obtained from $\mathcal{D}'$ by tensoring every module of morphisms by A. This category is still a pseudo quadratic Lie algebra but over A.

Let π be the projector $\pi = (\mathrm{Id} - T)/2$. It's an endomorphism of $L^{\otimes 2} = [2]$ and the corresponding module is $\bigwedge^2 L$. The trace of Id is δ^2, because it corresponds to two circles and the trace of T is δ. So we have:

$$\dim \bigwedge{}^2 L = \frac{\delta(\delta-1)}{2}$$

Consider the homomorphisms U and V of [2] corresponding to the following diagrams:





It is easy to see the following:

$$\pi U = U\pi = U \qquad\qquad U^2 = 2tU$$

Moreover U, V and π commutes. Since t is invertible there exists a projector π' such that: $U = 2t\pi'$. On the other hand Kneissler [K] has shown the following formula:

$$\omega(\pi V\pi)^2 = -\frac{3p\omega}{2}\pi V\pi + \Big(4t^3 - \frac{3\omega}{2}\Big)\Big(\frac{\omega t^2}{2} - \frac{3s\omega}{4}\Big)U$$

So there exists an element π'' in $\mathcal{D}'_A$ such that:

$$\pi V\pi = (4t^3 - \frac{3\omega}{2})\pi' - \frac{3p}{2}\pi''$$

where the element p is defined in A as the quotient $\dfrac{p\omega}{\omega}$. One can check that π'' is an idempotent commuting with π'. Let X_1, X_2 and E be the images of π', π'' and $\pi - \pi' - \pi''$. It is easy to compute the traces of U, V and $\pi V\pi$:

$$\tau(U) = 2t\delta \qquad\qquad \tau(V) = 8t^3\delta \qquad\qquad \tau(\pi V\pi) = \tau(\pi V) = 4t^3\delta$$

The result follows. □

Remark. Let L be a simple quadratic Lie (super)algebra over a ring k. Suppose that χ send t, ω and $p\omega$ to invertible elements in k. Suppose also that the dimension d of L is invertible in k. Then the functor Φ extends to a functor defined on $\mathcal{D}'_A$. One can check, case by case, that Φ send E to the zero module, and therefore the dimension d is given by: $d = 3 - \chi(\frac{2\omega}{p})$.

Actually the module E seems to be very poor. We don't have presently any counterexample to the following conjecture:

Conjecture. For every morphism u from [2] to [2], represented by a connected diagram, the induced morphism from E to E is trivial.

With regards to this conjecture, one mays expect to kill E in a suitable quotient of $\mathcal{D}'$ without loosing any important information. More precisely one has the following conjecture:

3.6 Conjecture. There exists a simple pseudo quadratic Lie algebra $\mathcal{L}$ over a ring Λ' and a morphism Φ from $\mathcal{D}'$ to $\mathcal{L}$ such that:

— the algebra Λ' is an integral domain contained in a localization of Λ

— if $\mathrm{Hom}_c([p],[q])$ is the module of homomorphisms in $\mathcal{D}'$ from an object $[p]$ to an object $[q]$ represented by connected diagrams, the functor Φ is injective on $\mathrm{Hom}_c([p],[q])$

— the module $\bigwedge^2 \mathcal{L}$ decomposes in a direct sum of two modules $X_1 \simeq \mathcal{L}$ and X_2, such that X_2 is absolutely simple (i.e. $\mathrm{End}(X_2)$ is contained in a localization of Λ)

Remark. If L is a simple quadratic Lie (super)algebra over a field k, the second exterior power $\bigwedge^2 L$ decomposes in a direct sum: $X_1 \oplus X_2$, where X_1 is isomorphic to L via the bracket, and X_2 is the kernel of the bracket. In many cases, X_1 and X_2 are simple. In the *sl* case, the module X_2 is not simple, but, in the subcategory of $\mathrm{Mod}(L)$ generated by L, the scalar product, the Casimir element, the bracket and the symmetry, it is simple. If L is the Lie superalgebra $\mathrm{D}(2,1,\alpha)$, X_2 is not simple, but the endomorphism ring of $\bigwedge^2 L$ is two-dimensional.

3.7 Theorem. *Suppose the conjecture 3.5 is true. Then there exist an extension Λ'' of Λ' and a decomposition in $\mathcal{L} \otimes \Lambda''$:*

$$\bigwedge{}^2\mathcal{L} = X_1 \oplus X_2$$

$$S^2\mathcal{L} = X_0 \oplus Y_2 \oplus Y_2' \oplus Y_2''$$

such that X_0, X_1, X_2, Y_2, Y_2' and Y_2'' are absolutely simple. Moreover there exists three elements α, β, γ in Λ'' such that: $t = \alpha + \beta + \gamma$, $s = \alpha\beta + \beta\gamma + \gamma\alpha$, $p = \alpha\beta\gamma$, and half the casimir operator acts on X_0, X_1, X_2, Y_2, Y_2' and Y_2'' by multiplication by 0, t, $2t$, $2t - \alpha$, $2t - \beta$ and $2t - \gamma$ respectively.

The dimension of these modules are the following:

$$dim X_0 = 1$$

$$dimX_1 = dimL = -\frac{(2t-\alpha)(2t-\beta)(2t-\gamma)}{\alpha\beta\gamma}$$

$$dimX_2 = \frac{(2t-\alpha)(2t-\beta)(2t-\gamma)(t+\alpha)(t+\beta)(t+\gamma)}{\alpha^2\beta^2\gamma^2}$$

$$dimY_2 = -\frac{t(2t-\beta)(2t-\gamma)(t+\beta)(t+\gamma)(3\alpha-2t)}{\alpha^2\beta\gamma(\alpha-\beta)(\alpha-\gamma)}$$

A Galois group $\mathfrak{S}_3$ acts by permuting the elements α, β and γ and the modules Y_2, Y_2' and Y_2''.

Sketch of proof. By assumption Λ is contained in an integral domain. Consider the algebra homomorphism $\varphi : R_0 \to \Lambda$. Suppose that φ is not injective. Let $P \in \mathbf{Q}[t,s,p]$ be an polynomial killed by φ, with $P \neq 0$. This polynomial has the following form: $P = t^n Q(t,s,p)$ with $Q(0,s,p) \neq 0$. Because t is not zero and Λ has no zero divisor, the polynomial Q is killed by φ. Consider the character from Λ to $\mathbf{Q}[s,p]$ associated with the Lie algebras of type $\mathrm{D}(2,1,\alpha)$. This character send Q to $Q(0,s,p)$ and Q cannot be zero.

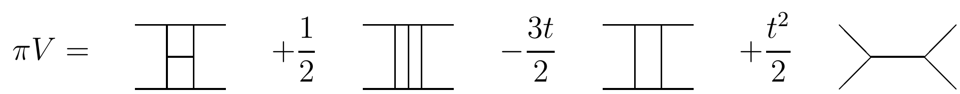
Thus R_0 is contained in $\Lambda \subset \Lambda'$. Up to inverting some elements in Λ and taking some algebraic extension, we may as well suppose that Λ'' contains $\mathbf{Q}[\alpha,\beta,\gamma]$ and every non zero homogenous element in $\mathbf{Q}[\alpha,\beta,\gamma]$ are invertible in Λ'.

Consider the diagram U and V defined in the proof of theorem 3.4. Since the functor is injective on every module of connected diagrams, there is no relations between U and V in $\mathcal{L}$, and projectors π' and π'' are non zero. Therefore π' generates X_1 and π'' generates X_2 and the projector π is the sum: $\pi' + \pi''$.

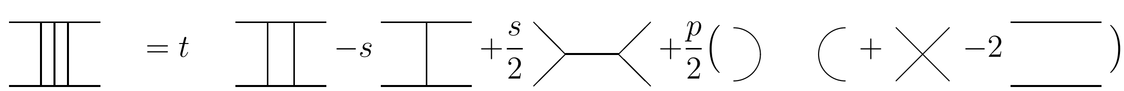
This relation may be written in the following way:

$$\pi V = (\frac{t^2}{2} - \frac{3s}{4})U - \frac{3p}{2}\pi$$

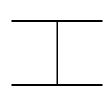
On the other hand it is easy to show the following:



If one substracts to this expression twice the same expression rotated by an half turn, one gets the following relation:



and that implies that the morphism ψ represented by



satisfies the following on the module $S^2\mathcal{L}$ divided by the image of the Casimir:

$$\psi^3 = t\psi^2 - s\psi + p$$

Hence the action of ψ on $S^2\mathcal{L}$ has three eigenspaces X_0, Y_2, Y_2' and Y_2'' corresponding to the eigenvalues $2t$, α, β and γ. On X_1 and X_2, ψ acts by multiplication by t and 0.

On the other hand the action oh half the Casimir on $\mathcal{L}\otimes\mathcal{L}$ is $2t-\psi$. So one gets the desired action.

The module $\mathrm{End}(\mathcal{L}^{\otimes 2}) = \mathrm{Hom}([2],[2])$ is isomorphic to the module $\mathrm{Hom}([0],[4])$ and the group $\mathfrak{S}_4$ acts on it. So we have the following decomposition:

$$\mathrm{Hom}([2],[2]) = E_+\otimes(4)\oplus E_-\otimes(1111)\oplus F_+\otimes(31)\oplus F_-\otimes(211)\oplus G\otimes(22)$$

where (4), (1111), ... are simple $\mathfrak{S}_4$-modules corresponding to Young diagrams. The assumption about the structure of $\bigwedge^2\mathcal{L}$ implies that G is isomorphic to the module $\mathrm{End}(X_1)\oplus\mathrm{End}(X_2)$ and that E_- and F_- are trivial modules. Since ψ has different eigenvalues in $S^2\mathcal{L}$ and $\bigwedge^2\mathcal{L}$, there is no homomorphism from $S^2\mathcal{L}$ to $\bigwedge^2\mathcal{L}$. Thus F_+ is zero too.

At the end we prove that E_+ is two dimensional and the dimension of the module of endomorphisms of $\mathcal{L}^{\otimes 2}$ is 6 (over some extension of Λ). The simplicity of the modules follows.

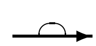
The computation of dimensions follows directly from:

$$\forall n>0 \qquad \tau(\psi^n) = 2tx_{n-1}\delta$$

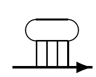
□

For the decomposition of $\mathcal{L}^{\otimes 3}$ the technique is much more complicated but we find a complete decomposition. In order to have absolutely simple modules we need to consider an algebraic extension of the ring. The first extension was necessary in order to have the modules Y_2, Y_2', Y_2''. This extension is the Galois extension of the polynomial X^3-tX^2+sX-p and the Galois group is $\mathfrak{S}_3$. This group permutes α, β and γ and permutes somes modules. In order to have a complete decomposition of $\mathcal{L}^{\otimes 3}$, one needs another Galois extension with Galois group G still isomorphic to $\mathfrak{S}_3$.

In order to detect some module, one need another operator. The first operator ψ was strongly related to the Casimir operator. Denote by π half this operator. Actually every element in the algebra $\mathcal{A}(S^1)$ induces an operator on every module. The Casimir operator 2π is obtained from the diagram:



Consider the operator π' represented by the diagram:



an set: $\sigma = \pi' - (8t^3-3\omega)\pi$. This element acts on every module and in partcular on every direct summand of $\mathcal{L}^{\otimes n}$. On an absolutely simple module it acts by a scalar.

In order to describe the decomposition of $\mathcal{L}^{\otimes 3}$, we will use the standard action of $\mathfrak{S}_3$ on $\mathcal{L}^{\otimes 3}$. For every Young diagram a there is a corresponding module $(a)\mathcal{L}$. In this case we have the following modules $(3)\mathcal{L}$, $(21)\mathcal{L}$ and $(111)\mathcal{L}$. The first one is the symmetric power $S^3\mathcal{L}$ and the last one is the exterior power $\bigwedge^3 \mathcal{L}$.

Now we need to define another cubic extension of the ring. Consider the following elements in Λ':

$$p = \alpha\beta\gamma \quad q = t(\alpha\beta + \beta\gamma + \gamma\alpha) \quad r = t^3$$

Set also:

$$a = -\frac{9p}{8} + \frac{q}{4} - \frac{r}{2}$$

$$b = -\frac{27p^2}{32} + \frac{15pq}{16} - \frac{pr}{8} - \frac{q^2}{2}$$

$$c = \frac{p^2}{64}(27p - 18q + 4r)$$

Now define Λ'' as the Galois extension corresponding to the polynomial: $\Pi = X^3 - aX^2 + bX - c$. In this extension Π has three roots λ, μ and ν and the Galois group G is isomorphic to $\mathfrak{S}_3$. The Galois group of the complete extension is isomorphic to $\mathfrak{S}_3 \times \mathfrak{S}_3$.

3.8 Theorem. *Suppose the conjecture is true. Then in some localization of Λ'', $\mathcal{L}^{\otimes 3}$ has the following decomposition in absolutely simple modules:*

— $(3)\mathcal{L} = S^3\mathcal{L} = 2X_1 \oplus X_2 \oplus B \oplus B' \oplus B'' \oplus Y_3 \oplus Y_3' \oplus Y_3''$

— $(21)\mathcal{L} = 2X_1 \oplus 2X_2 \oplus Y_2 \oplus Y_2' \oplus Y_2'' \oplus B \oplus B' \oplus B'' \oplus C \oplus C' \oplus C''$

—$(111)\mathcal{L} = X_0 \oplus X_2 \oplus Y_2 \oplus Y_2' \oplus Y_2'' \oplus X_3 \oplus X_3' \oplus X_3''$

— *one has also the following decomposition:*

$X_1 \otimes Y_2 = X_1 \oplus X_2 \oplus Y_2 \oplus Y_3 \oplus B' \oplus B'' \oplus C$

$X_1 \otimes X_2 = X_1 \oplus 2X_2 \oplus Y_2 \oplus Y_2' \oplus Y_2'' \oplus B \oplus B' \oplus B'' \oplus C \oplus C' \oplus C'' \oplus X_3 \oplus X_3' \oplus X_3''$

— *the Galois group $\mathfrak{S}_3$ permutes α, β and γ and G permutes λ, μ, ν.*

— *$\mathfrak{S}_3$ permutes X, X'' and X'' for $X = Y_2$, Y_3, B or C and G permutes X_3, X_3' and X_3''*

— *the actions of π and σ are the following:*

X_0 :	$\pi = 0$	$\sigma = 0$
X_1 :	$\pi = t$	$\sigma = 0$
X_2 :	$\pi = 2t$	$\sigma = -18pt$
Y_2 :	$\pi = 2t - \alpha$	$\sigma = 2\alpha(\alpha - t)(\alpha - 2t)(3\alpha - t)$
X_3 :	$\pi = 3t$	$\sigma = -6t(9p + 4\lambda)$
Y_3 :	$\pi = 3t - 3\alpha$	$\sigma = 6\alpha(\alpha - t)(3\alpha - t)(3\alpha - 2t)$
B :	$\pi = 2t + \alpha$	$\sigma = 2\alpha(\alpha + t)((\beta + \gamma)(2\beta + 2\gamma - \alpha) - 12\beta\gamma)$
C :	$\pi = 3t - 3\alpha/2$	$\sigma = 3\alpha(\alpha - 2t)(t^2 - 9s/2 + 9\beta\gamma)$

— *the dimensions are the following:*

$$dim Y_3 = \frac{1}{3}\frac{t(t+\beta)(t+\gamma)(2t-\alpha)(2t-\beta)(2t-\gamma)(5\alpha-2t)(2\beta+\gamma)(2\gamma+\beta)}{\alpha^3\beta\gamma(2\alpha-\beta)(2\alpha-\gamma)(\alpha-\beta)(\alpha-\gamma)}$$

$$dimB = -\frac{t(t+\beta)(t+\gamma)(2t-\alpha)(2t-\beta)(2t-\gamma)(2t-3\beta)(2t-3\gamma)(2\alpha+\beta)(2\alpha+\gamma)}{\alpha^2\beta^2\gamma^2(\alpha-\beta)(\alpha-\gamma)(2\beta-\gamma)(2\gamma-\beta)}$$

$$dimC = -\frac{32}{3}\frac{t(t+\alpha)(t+\beta)(t+\gamma)(2t-\beta)(2t-\gamma)(\beta+\gamma)(\beta+2\gamma)(\gamma+2\beta)}{\alpha^3\beta\gamma(\alpha-2\beta)(\alpha-2\gamma)(\alpha-\beta)(\alpha-\gamma)}$$

$$dimX_3 = \frac{d(27p-18q+4r)}{12\lambda(\lambda-\mu)(\lambda-\nu)}\Big(\frac{1}{16}(q+2r)(7p+2q+4r)-\lambda(\mu+\nu)+\lambda(-\frac{3}{4}p+\frac{3}{2}q+r)\Big)$$

where $d = dimX_1$ *is the dimension of* $\mathcal{L}$*.*

Remark. The computation is rather difficult. The program maple is very usefull for that. Actually this decomposition holds for every simple quadratic Lie (super)algebra. But sometime, some of these modules are zero. Another possibility is that the sum of two modules is zero.

In the sl case, we have $\alpha = t$. The polynomial Π has roots $\lambda = -p/4$ and $\mu = p/2$ and: $C = X_3'' = 0$.

In the osp case, we have $\beta+2\gamma = 0$. We have: $\lambda = 3p/4+t\gamma^2$ and $\mu = -3p/2-2t\gamma^2$ and: $Y_3 = C = B'' = X_3'' = 0$.

In the exceptional cases we have $3\alpha = 2t$ and $\lambda = 0$ and:

$$B' = B'' = Y_2 = C \oplus X_2 = Y_3 \oplus X_1 = X_3' = X_3'' = 0$$

If $\alpha + t = 0$, we get the sl_2 case and we have: $\lambda = -2t^3 - 9p/4$ and:

$$X_2 = Y_2' = Y_2'' = Y_3' = Y_3'' = B' = B'' = C = C' = C'' = X_3'' = 0$$

$$B + X_1 = Y_2 + X_3 = 0$$

In the D(2, 1,?) case, we have $t = 0$ and $\lambda = 3p/4$, $\mu = -3p/2$ and all the modules are zero dimensional except X_0, X_1, X_2, X_3 and X_3'.

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