11 Appendix. Examples of [21]-colored HOMFLY

Here we list examples of knots from the Rolfsen table of [21], for which the [21]-colored HOMFLY are now available.

11.1 A list of knots with up to 10 intersections

We start from the table which contains some relevant information about the knots up to 10 intersections.

The left part of the table lists the previously known cases:

- For the torus knots, the arbitrary colored HOMFLY polynomial is given by the Rosso-Jones formula [32].
- For *three*-strand knots, the [21]-colored HOMFLY polynomials can be calculated by the cabling method of [16] on ordinary computers. When the number of strands is three, we explicitly give a braid word, only instead of $\tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\dots$ we write the sequence $a_1, a_2, a_3, \dots$
- For *two*-bridge knots the knowledge of Racah matrices S and $\bar{S}$ from [17] is sufficient: eq.(2) in this case reduces to an obvious matrix element $\mathcal{A}_{11}$. The corresponding answers are available from [17]. When the number of bridges is two, we explicitly give an $S-T$ word.

The right part of the table lists the cases which are available only now, by the method of the present paper, these knots are boldfaced in the first column. Numerous intersections between the left and right parts of the table are important for checking our conjecture (2).

An exhaustive list of the pretzel knots up to 10 intersections is borrowed from the third paper of ref.[20]. They are distinguished from generic double fat tree knots only by simplicity of computer calculations, what is actually quite important.

The starfish cases are also usually simple enough: they still involve just one sum over representations (all pretzel knots are automatically starfish). The cases with two and three propagators involve two and three such sums and are considerably more difficult for computers. A search for maximally simple representations are therefore important from this point of view, and hopefully the table can be significantly improved.

The right part of the table is currently incomplete, but it seems all the knots with 10 or less intersections to fit into it (with the possible exception of 10_{161}, which is anyhow present in the left part of the table).

In addition to this table, answers are available for some explicitly identified mutants with 11 intersections.

				Implications of the new topological theory (2)		
knot	torus	# of strands	# of bridges	pretzel	starfish	double-sum
3_1	$[2,3]$	$(1,1)^2$	$d_R\cdot(ST^3S^\dagger)_{11}$	$(3,0)$		
4_1		$(1,-1)^2$	$d_R\cdot(\bar S\bar T^{-2}\bar S\bar T^2\bar S)_{11}$	$(1,\bar 2,1)$		
5_1	$[2,5]$	2	$d_R\cdot(ST^5S^\dagger)_{11}$	$(5,0)$		
5_2		$(3,1,-1,1)$	$d_R\cdot(\bar S\bar T^4\bar S\bar T^2\bar S)_{11}$	$(\bar 3,\bar 1,\bar 1)$		
6_1		4	$d_R\cdot(\bar S\bar T^{-4}\bar S\bar T^2\bar S)_{11}$	$(\bar 5,-\bar 1,-\bar 1)$		
6_2		$(3,-1,1,-1)$	$d_R\cdot(\bar S\bar T^2\bar S\bar T^{-1}ST^2S^\dagger\bar T^{-1}\bar S)_{11}$	$(3,\bar 2,1)$		
6_3		$(2,-1,1,-2)$	$d_R\cdot(ST^{-2}S^\dagger\bar T\bar S\bar T^{-1}ST^2S^\dagger)_{11}$	$(2,-3,1,1)$		
7_1	$[2,7]$	2	$d_R\cdot(ST^7S^\dagger)_{11}$	$(7,0)$		
7_2		4	$d_R\cdot(\bar S\bar T^6\bar S\bar T^2\bar S)_{11}$	$(\bar 5,\bar 1,\bar 1)$		
7_3		$(5,1,-1,1)$	$d_R\cdot(ST^4S^\dagger\bar T^{-3}\bar S)_{11}$	$(4,1,1,1)$		
7_4		4	$d_R\cdot(\bar S\bar T^{-3}STS^\dagger\bar T^{-2}STS^\dagger)_{11}$	$(\bar 3,\bar 3,\bar 1)$		
7_5		$(4,1,-1,2)$	$d_R\cdot(ST^3S^\dagger\bar T^{-2}STS^\dagger\bar T^{-1}\bar S)_{11}$	$(3,2,1,1)$		
7_6		4	$d_R\cdot(\bar S\bar T^2\bar S\bar T^{-2}\bar S\bar TST^{-2}S^\dagger)_{11}$	$(-3,1,\bar 2,1,1)$		
7_7		4	$d_R\cdot(\bar S\bar T^{-2}\bar S\bar TST^{-1}S^\dagger\bar T\bar S\bar T^{-2}\bar S)_{11}$	$(-\bar 3,\bar 1,-\bar 3,\bar 1,\bar 1)$		
8_1		5	$d_R\cdot(\bar S\bar T^{-6}\bar S\bar T^2\bar S)_{11}$	$(1,\bar 6,1)$		
8_2		$5,-1,1,-1$	$d_R\cdot(STS^\dagger\bar T^{-1}\bar S\bar TST^{-5}S^\dagger)_{11}$	$(5,\bar 2,1)$		
8_3		5	$d_R\cdot(\bar S\bar T^4\bar S\bar T^{-4}\bar S)_{11}$	$(1,1,\bar 4,1,1)$		
8_4		4	$d_R\cdot(\bar S\bar T^4\bar S\bar T^{-1}ST^3S^\dagger)_{11}$	$(3,\bar 4,1)$		
8_5		$3,-1,3,-1$	3	$(3,\bar 2,3)$		
8_6		4	$d_R\cdot(STS^\dagger\bar T^{-1}\bar S\bar T^3ST^{-3}S^\dagger)_{11}$	$(1,3,\bar 2,1,1)$		
8_7		$4,-1,1,-2$	$d_R\cdot(ST^2S^\dagger\bar T^{-1}\bar S\bar TST^{-4}S^\dagger)_{11}$	$(4,-3,1,1)$		
8_8		4	$d_R\cdot(\bar S\bar TST^{-1}S^\dagger\bar T\bar S\bar T^{-3}ST^2S^\dagger)_{11}$	$(2,-3,1,1,1,1)$		
8_9		$3,-1,1,-3$	$d_R\cdot(ST^{-3}S^\dagger\bar T\bar S\bar T^{-1}ST^3S^\dagger)_{11}$	$(4,-3,-1,-1)$		
8_{10}		$3,-1,2,-2$	3	$(2,-3,1,3)$		
8_{11}		4	$d_R\cdot(STS^\dagger\bar T^{-1}\bar S\bar TST^{-2}S^\dagger\bar T^3\bar S)_{11}$	$(-\bar 3,\bar 1,\bar 1,\bar 3,\bar 1)$		
8_{12}		5	$d_R\cdot(\bar S\bar T^2\bar S\bar T^{-2}\bar S\bar T^2\bar S\bar T^{-2}\bar S)_{11}$			
8_{13}		4	$d_R\cdot(\bar S\bar T^3ST^{-1}S^\dagger\bar T\bar S\bar T^{-1}STS^\dagger\bar T^{-1}\bar S)_{11}$	$(-\bar 4,-3,1,1,1)$		
8_{14}		4	$d_R\cdot(ST^3S^\dagger\bar T\bar S\bar T^{-1}STS^\dagger\bar T^{-1}\bar S\bar T^2\bar S)_{11}$			
$\mathbf{8_{15}}$		4	3	$(2,3,3,-1,-1,-1)$	(133)	
8_{16}		$2,-1,2,-1,1,-1$	3			
8_{17}		$2,-1,1,-1,1,-2$	3			
8_{18}		$1,-1,1,-1,1,-1,1,-1$	3			
8_{19}	$[3,4]$	$3,1,3,1$	3	$(3,-\bar 2,3)$		
8_{20}		$3,-1,-3,-1$	3	$(3,\bar 2,-3)$		
8_{21}		$3,-1,-3,-1$	3	$(2,-3,1,-3)$		

9_1	$[2,9]$	$(9,0)$	$d_R \cdot (ST^9S^\dagger)_{11}$			
9_2		5	$d_R \cdot (\bar{S}\bar{T}^8\bar{S}\bar{T}^2\bar{S})_{11}$	$(\bar{1},\bar{7},\bar{1})$		
9_3		$7,1,-1,1$	$d_R \cdot (\bar{S}\bar{T}^{-3}ST^6S^\dagger)_{11}$	$(6,1,1,1)$		
9_4		4	$d_R \cdot (ST^{-1}S^\dagger\bar{T}^4ST^{-4}S^\dagger)_{11}$	$(4,1^5)$		
9_5		5	$d_R \cdot (\bar{S}\bar{T}^5ST^{-1}S^\dagger\bar{T}^2ST^{-1}S^\dagger)_{11}$	$(-\bar{1},-\bar{3},-\bar{5})$		
9_6		$6,1,-1,2$	$d_R \cdot (\bar{S}\bar{T}^{-1}STS^\dagger\bar{T}^{-2}ST^5S^\dagger)_{11}$	$(2,1,5,1)$		
9_7		4	$d_R \cdot (\bar{S}\bar{T}^{-1}STS^\dagger\bar{T}^{-4}ST^3S^\dagger)_{11}$	$(2,3,1^4)$		
9_8		5	$d_R \cdot (\bar{S}\bar{T}^{-2}\bar{S}\bar{T}^4\bar{S}\bar{T}^{-1}STS^\dagger\bar{T}^{-1}\bar{S})_{11}$	$(-2,-3,1^6)$		
9_9		$5,1,-1,3$	$d_R \cdot (\bar{S}\bar{T}^{-1}ST^2S^\dagger\bar{T}^{-2}ST^4S^\dagger)_{11}$	$(-4,1,-5,1)$		
9_{10}		4	$d_R \cdot (\bar{S}\bar{T}^3ST^{-3}S^\dagger\bar{T}^{-2}ST^{-1}S^\dagger)_{11}$	$(\bar{3},\bar{3},\bar{1},\bar{1},\bar{1})$		
9_{11}		4	$d_R \cdot (\bar{S}\bar{T}^{-2}\bar{S}\bar{T}^2\bar{S}\bar{T}^{-1}ST^4S^\dagger)_{11}$	$(-5,-2,1^4)$		
9_{12}		5	$d_R \cdot (\bar{S}\bar{T}^4\bar{S}\bar{T}^{-2}\bar{S}\bar{T}ST^{-1}S^\dagger\bar{T}\bar{S})_{11}$	$(-3,1,1,1,\bar{4})$		
9_{13}		4	$d_R \cdot (ST^3S^\dagger\bar{T}^{-2}STS^\dagger\bar{T}^{-3}\bar{S})_{11}$	$(1,3,1,1,-\bar{4})$		
9_{14}		5	2	$(-\bar{5},-\bar{3},\bar{1},\bar{1},\bar{1})$		
9_{15}		5	2			
9_{16}		$4,2,-1,3$	3	$(2,1,3,3)$		
9_{17}		4	2	$(\bar{3},\bar{3},-\bar{1}^5)$		
9_{18}		4	2			
9_{19}		5	2			
9_{20}		4	2	$(4,3,-1^4)$		
9_{21}		5	2			
$\mathbf{9_{22}}$		4	3			
9_{23}		4	2			
$\mathbf{9_{24}}$		4	3	$(-2,-3,3,1^3)$		
$\mathbf{9_{25}}$		5	3			
9_{26}		4	2			
9_{27}		4	2			
$\mathbf{9_{28}}$		4	3	$(2,-3,-3,1^3)$		
$\mathbf{9_{29}}$		4	3			(131)
$\mathbf{9_{30}}$		4	3			
9_{31}		4	2			
$\mathbf{9_{32}}$		4	3			
$\mathbf{9_{33}}$		4	3			
$\mathbf{9_{34}}$		4	3			
$\mathbf{9_{35}}$		5	3	$(\bar{3},\bar{3},\bar{3})$		
$\mathbf{9_{36}}$		4	3			
$\mathbf{9_{37}}$		5	3	$(-\bar{3},-\bar{3},\bar{3},\bar{1},\bar{1})$		
$\mathbf{9_{38}}$		4	3			
$\mathbf{9_{39}}$		5	3			
$\mathbf{9_{40}}$		4	3			
$\mathbf{9_{41}}$		5	3			
$\mathbf{9_{42}}$		4	3		$\overline{\sum_{X\in R\otimes\bar{R}}} \frac{d_R \cdot (ST^3S^\dagger)_{1X}(ST^{-2}S^\dagger)_{1X}(STS^\dagger\bar{T}^{-1}\bar{S}\bar{T}^2\bar{S})_{1X}}{\bar{S}_{1X}}$	
$\mathbf{9_{43}}$		4	3			
$\mathbf{9_{44}}$		4	3			
$\mathbf{9_{45}}$		4	3			
$\mathbf{9_{46}}$		4	3	$(\bar{3},-\bar{3},\bar{3})$		
$\mathbf{9_{47}}$		4	3			
$\mathbf{9_{48}}$		4	3	$(-\bar{3},-\bar{3},-\bar{3},\bar{1},\bar{1})$		
$\mathbf{9_{49}}$		4	3			

10_1			$(\bar{S}\bar{T}^{-8}\bar{S}\bar{T}^{2}\bar{S})_{11}$	$(\bar{1},\bar{7},-\bar{3})$			
10_2		$7,-1,1,-1$	2	$(2,-7,-1,-1)$			
10_3			2	$(\bar{1},\bar{5},-\bar{5})$			
10_4			2	$(-\bar{7},\bar{1},\bar{1},\bar{1},\bar{1})$			
10_5		$6,-1,1,-2$	2	$(-2,7,-1,-1)$			
10_6			2	$(-5,-1,-1,-1,-\bar{2})$			
10_7			2	$(-\bar{3},\bar{1},\bar{5},\bar{1},\bar{1})$			
10_8			2	$(-6,1,1,1,1,1)$			
10_9		$5,-1,1,-3$	2	$(6,-3,-1,-1)$			
10_{10}			2				
10_{11}			2	$(3,1,1,1,\bar{4})$			
10_{12}			2	$(4,-3,1,1,1,1)$			
10_{13}			2				
10_{14}			2				
10_{15}			2	$(-2,-1,5,-1,-1,-1)$			
10_{16}			2	$(-\bar{3},-\bar{1},\bar{5},-\bar{1},-\bar{1})$			
10_{17}		$4,-1,1,-4$	2	$(4,-5,1,1)$			
10_{18}			2				
10_{19}			2	$(\bar{4},5,-1,-1,-1)$			
10_{20}			2	$(-2,1,3,1,1,1,1,1)$			
10_{21}			2	$(-\bar{3},\bar{1},\bar{1},\bar{3},\bar{1},\bar{1},\bar{1})$			
10_{22}			2	$(-4,1,1,3,1,1)$			
10_{23}			2				
10_{24}			2				
10_{25}			2				
10_{26}			2				
10_{27}			2				
10_{28}			2	$(\bar{4},3,-1,-1,-1,-1,-1)$			
10_{29}			2				
10_{30}			2				
10_{31}			2				
10_{32}			2				
10_{33}			2				
10_{34}			2	$(2,-3,1,1,1,1,1,1)$			
10_{35}			2				
10_{36}			2				
10_{37}			2				
10_{38}			2				
10_{39}			2				
10_{40}			2				
10_{41}			2				
10_{42}			2				
10_{43}			2				
10_{44}			2				
10_{45}			2				
10_{46}		$5,-1,3,-1$	3	$(-2,3,5,1)$ or $(\bar{2},5,3)$			
10_{47}		$5,-1,2,-2$	3	$(2,-3,5,1)$			
10_{48}		$4,-2,1,-3$	3	$(2,-5,1,3)$			
$\mathbf{10_{49}}$			3	$(2,3,5,-1,-1,-1)$ or $(\bar{2},-5,-3,1,1)$			
$\mathbf{10_{50}}$			3				
$\mathbf{10_{51}}$			3				
$\mathbf{10_{52}}$			3				
$\mathbf{10_{53}}$			3				
$\mathbf{10_{54}}$			3				
$\mathbf{10_{55}}$			3				

$\mathbf{10_{56}}$			3				
$\mathbf{10_{57}}$			3				
$\mathbf{10_{58}}$			3				
$\mathbf{10_{59}}$			3				
$\mathbf{10_{60}}$			3				
$\mathbf{10_{61}}$			3	$(3,3,\bar{4})$			
10_{62}		$4,-1,3,-2$	3	$(4,-3,1,3)$			
$\mathbf{10_{63}}$			3	$(\bar{4},-3,-3,1,1)$			
10_{64}		$3,-1,3,-3$	3	$(-4,3,3,1)$			
$\mathbf{10_{65}}$			3	$(\bar{4},3,-3,-1,-1)$			
$\mathbf{10_{66}}$			3	$(4,3,3,-1,-1,-1)$			
$\mathbf{10_{67}}$			3				
$\mathbf{10_{68}}$			3				
$\mathbf{10_{69}}$			3				
$\mathbf{10_{70}}$			3				
$\mathbf{10_{71}}$			3		$\overline{\sum_{X\in R\otimes\bar{R}} \frac{d_R\cdot(\bar{S}\bar{T}^2\bar{S}\bar{T}^{-2}\bar{S})_{1X}\,(ST^2S^{\dagger}\bar{T}^{-1}\bar{S})_{1X}\,(ST^{-3}S^{\dagger}\bar{T}^{-1}\bar{S})_{1X}}{\bar{S}_{1X}}}$		
$\mathbf{10_{72}}$			3				
$\mathbf{10_{73}}$			3				
$\mathbf{10_{74}}$			3	$(-\bar{3},\bar{1},\bar{3},\bar{3},\bar{1})$			
$\mathbf{10_{75}}$			3				
$\mathbf{10_{76}}$			3	$(1,3,3,1,\bar{2})$			
$\mathbf{10_{77}}$			3	$(2,-3,1,3,1,1)$			
$\mathbf{10_{78}}$			3	$(\bar{2},-3,-3,1,1,1,1)$			
10_{79}		$3,-2,2,-3$	3				
$\mathbf{10_{80}}$			3				
$\mathbf{10_{81}}$			3				
10_{82}		$4,-1,1,-1,1,-2$	3				
$\mathbf{10_{83}}$			3				
$\mathbf{10_{84}}$			3				
10_{85}		$4,-1,2,-1,1,-1$	3				
$\mathbf{10_{86}}$			3				
$\mathbf{10_{87}}$			3				
$\mathbf{10_{88}}$			3				
$\mathbf{10_{89}}$			3				
$\mathbf{10_{90}}$			3				
10_{91}		$3,-1,1,-2,1,-2$	3				
$\mathbf{10_{92}}$			3				
$\mathbf{10_{93}}$			3				
10_{94}		$3,-1,2,-2,1,-1$	3				
$\mathbf{10_{95}}$			3				
$\mathbf{10_{96}}$			3				
$\mathbf{10_{97}}$			3				
$\mathbf{10_{98}}$			3				
10_{99}		$2,-1,2,-2,1,-2$	3				
10_{100}		$3,-1,2,-1,2,-1$	3				
$\mathbf{10_{101}}$			3				
$\mathbf{10_{102}}$			3				
$\mathbf{10_{103}}$			3				
10_{104}		$3,-2,1,-1,1,-2$	3				
$\mathbf{10_{105}}$			3				
10_{106}		$3,-1,1,-1,2,-2$	3				
$\mathbf{10_{107}}$			3				
$\mathbf{10_{108}}$			3				
10_{109}		$2,-1,1,-2,2,-2$	3				
$\mathbf{10_{110}}$			3				

$\mathbf{10_{111}}$			3				
10_{112}		$3,-1,1,-1,1,-1,1,-1$	3				
$\mathbf{10_{113}}$			3				
$\mathbf{10_{114}}$			3				
$\mathbf{10_{115}}$			3				
10_{116}		$2,-1,2,-1,1,-1,1,-1$	3				
$\mathbf{10_{117}}$			3				
$\mathbf{10_{118}}$			3				
$\mathbf{10_{119}}$			3				
$\mathbf{10_{120}}$			3				
$\mathbf{10_{121}}$			3				
$\mathbf{10_{122}}$			3				
10_{123}		$(1,-1)^5$	3				
10_{124}	$[3,5]$	$5,1,3,1$	3	$(2,-1,5,3)$ or $(\bar{2},-5,-3)$	$\overline{\sum_{X\in R\otimes\bar{R}}}\frac{d_R\cdot(ST^5S^\dagger)_{1X}(ST^2S^\dagger)_{1X}(ST^2S^\dagger\bar{T}^{-1}\bar{S})_{1X}}{\bar{S}_{1X}}$		
10_{125}		$5,-1,-3,-1$	3	$(2,-5,-1,3)$ or $(\bar{2},5,-3)$			
10_{126}		$5,1,-3,1$	3	$(-2,3,-5,1)$ or $(\bar{2},-5,3)$			
10_{127}		$5,1,-2,2$	3	$(2,-5,-3,1)$			
$\mathbf{10_{128}}$			3		$\overline{\sum_{X\in R\otimes\bar{R}}}\frac{d_R\cdot(\bar{S}\bar{T}^{-2}\bar{S})_{1X}(\bar{S}\bar{T}^{-1}ST^2S^\dagger)_{1X}(\bar{S}\bar{T}^{-3}ST^2S^\dagger)_{1X}}{\bar{S}_{1X}}$		
$\mathbf{10_{129}}$			3	$(2,1,1,-3,1,1)$			
$\mathbf{10_{130}}$			3				
$\mathbf{10_{131}}$			3				
$\mathbf{10_{132}}$			3		$\overline{\sum_{X\in R\otimes\bar{R}}}\frac{d_R\cdot(ST^{-2}S^\dagger\bar{T}^3\bar{S})_{1X}(ST^{-2}S^\dagger)_{1X}(\bar{S}\bar{T}^{-1}ST^2S^\dagger)_{1X}}{\bar{S}_{1X}}$		
$\mathbf{10_{133}}$			3				
$\mathbf{10_{134}}$			3				
$\mathbf{10_{135}}$			3				
$\mathbf{10_{136}}$			3		$\overline{\sum_{X\in R\otimes\bar{R}}}\frac{d_R\cdot(\bar{S}\bar{T}^{-2}\bar{S}\bar{T}^2\bar{S})_{1X}(ST^2S^\dagger)_{1X}(\bar{S}\bar{T}^{-3}ST^{-2}S^\dagger)_{1X}}{\bar{S}_{1X}}$		
$\mathbf{10_{137}}$			3				
$\mathbf{10_{138}}$			3				
10_{139}		$4,1,3,2$	3	$(4,-1,3,3)$			
$\mathbf{10_{140}}$			3	$(-3,3,\bar{4})$			
10_{141}		$4,-1,-3,-2$	3	$(4,-3,-3,1)$			
$\mathbf{10_{142}}$			3	$(3,3,-\bar{4})$			
10_{143}		$4,1,-3,2$	3	$(-4,3,1,-3)$			
$\mathbf{10_{144}}$			3	$(\bar{4},3,3,-1,-1)$			
$\mathbf{10_{145}}$			3		$\sum_{X\in R\otimes R}\frac{d_R\cdot(ST^{-2}S^\dagger\bar{T}^2S)_{1X}(\bar{S}\bar{T}^3S)_{1X}(ST^{-1}S^\dagger\bar{T}\bar{S}\bar{T}^{-1}S)_{1X}}{S_{1X}}$		
$\mathbf{10_{146}}$			3				
$\mathbf{10_{147}}$			3				
10_{148}		$4,1,-2,1,-1,1$	3				
10_{149}		$4,1,-1,1,-1,2$	3				
$\mathbf{10_{150}}$			3				
$\mathbf{10_{151}}$			3				
10_{152}		$3,2,2,3$	3			(163)	
$\mathbf{10_{153}}$			3			(165)	
$\mathbf{10_{154}}$			3			(167)	
10_{155}		$3,1,-2,1,-2,1$	3				
$\mathbf{10_{156}}$			3				
10_{157}		$3,2,-1,1,-1,2$	3				
$\mathbf{10_{158}}$			3				
10_{159}		$3,1,-1,1,-2,2$	3				
$\mathbf{10_{160}}$			3				
10_{161}		$3,1,-1,1,2,2$	3				
$\mathbf{10_{162}}$			3				
$\mathbf{10_{163}}$			3				
$\mathbf{10_{164}}$			3				
$\mathbf{10_{165}}$			3				

$$9_{29}: \qquad \sum_{i,j} \frac{A_1[1,i]A_2[1,i]S[j,i]A_3[1,j]A_4[1,j]}{\bar{S}[1,i]S[1,j]} \tag{131}$$

where

$$\begin{aligned} A_1 &= (ST^{-3}S^t) \\ A_2 &= (\bar{S}\bar{T}^2\bar{S}) \\ A_3 &= (\bar{S}\bar{T}^{-3}S) \\ A_4 &= (\bar{S}\bar{T}^{-2}\bar{S}\bar{T}S) \end{aligned} \tag{132}$$

11.2 Some thin knots

A lot of explicit examples with less than nine intersections can be found in [16] and [17]. Here we add the only 8-intersection knot 8_{15} which was not present in those lists (it is pretzel, but possesses also a simpler triple-finger realization),

$$H_R^{8_{15}} = Pr(-2,-3,-3,1,1,1) = \overline{\sum_{X\in R\otimes\bar{R}}} \frac{d_R}{\bar{S}_{1X}} (\bar{S}\bar{T}^2\bar{S})_{1X}(ST^{-3}S^\dagger)_{1X}(ST^{-2}S^\dagger\bar{T}^1\bar{S}\bar{T}^{-1})_{1X} \tag{133}$$

and a couple of 10-intersection knots:

$$H_R^{10_{71}} = \overline{\sum_{X\in R\otimes\bar{R}}} \frac{d_R}{\bar{S}_{1X}} (\bar{S}\bar{T}^2\bar{S}\bar{T}^{-2}\bar{S})_{1X}(ST^2S^\dagger\bar{T}^{-1}\bar{S})_{1X}(ST^{-3}S^\dagger\bar{T}^{-1}\bar{S})_{1X} \tag{134}$$

and

$$H_R^{10_{125}} = \overline{\sum_{X\in R\otimes\bar{R}}} \frac{d_R}{\bar{S}^2_{1X}} (ST^{-5}S^\dagger)_{1X}(ST^{-1}S^\dagger)_{1X}(ST^3S^\dagger)_{1X}(ST^2S^\dagger)_{1X} \tag{135}$$

$$H^{8_{15}}_{[21]} = \frac{1}{q^{24}A^{30}} \cdot \tag{136}$$

$$\begin{pmatrix}
0 & 0 & 1 & -4 & 10 & -20 & 35 & -54 & 76 & -98 & 117 & -130 & 135 & -130 & 117 & -98 & 76 & -54 & 35 & -20 & 10 & -4 & 1 & 0 & 0 \\
0 & 2 & -8 & 26 & -61 & 117 & -192 & 291 & -402 & 510 & -598 & 662 & -685 & 662 & -598 & 510 & -402 & 291 & -192 & 117 & -61 & 26 & -8 & 2 & 0 \\
3 & -11 & 37 & -85 & 163 & -275 & 430 & -609 & 795 & -975 & 1133 & -1228 & 1259 & -1228 & 1133 & -975 & 795 & -609 & 430 & -275 & 163 & -85 & 37 & -11 & 3 \\
-5 & 16 & -45 & 87 & -159 & 256 & -382 & 516 & -672 & 808 & -919 & 992 & -1028 & 992 & -919 & 808 & -672 & 516 & -382 & 256 & -159 & 87 & -45 & 16 & -5 \\
2 & -8 & 15 & -28 & 50 & -78 & 105 & -143 & 170 & -203 & 219 & -239 & 234 & -239 & 219 & -203 & 170 & -143 & 105 & -78 & 50 & -28 & 15 & -8 & 2 \\
0 & 1 & 0 & 5 & 2 & 0 & 17 & -17 & 56 & -59 & 94 & -87 & 123 & -87 & 94 & -59 & 56 & -17 & 17 & 0 & 2 & 5 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & -5 & 0 & -13 & 10 & -25 & 8 & -42 & 19 & -35 & 19 & -42 & 8 & -25 & 10 & -13 & 0 & -5 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3 & 6 & 2 & 4 & -2 & 13 & 5 & 13 & -2 & 4 & 2 & 6 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -2 & -2 & 2 & -2 & -2 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$H^{10_{71}}_{[21]} = \frac{1}{q^{30}A^{12}} \cdot \tag{137}$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -3 & 4 & -5 & 5 & -5 & 4 & -3 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -3 & 5 & -8 & 13 & -15 & 20 & -21 & 23 & -21 & 20 & -15 & 13 & -8 & 5 & -3 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 4 & -11 & 20 & -39 & 56 & -84 & 104 & -131 & 140 & -150 & 140 & -131 & 104 & -84 & 56 & -39 & 20 & -11 & 4 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & -5 & 12 & -18 & 27 & -26 & 27 & -4 & -18 & 70 & -105 & 155 & -175 & 206 & -175 & 155 & -105 & 70 & -18 & -4 & 27 & -26 & 27 & -18 & 12 & -5 & 2 & 0 & 0 \\
-1 & 4 & -16 & 40 & -89 & 164 & -290 & 451 & -681 & 938 & -1251 & 1545 & -1865 & 2085 & -2267 & 2301 & -2267 & 2085 & -1865 & 1545 & -1251 & 938 & -681 & 451 & -290 & 164 & -89 & 40 & -16 & 4 & -1 \\
4 & -18 & 59 & -144 & 315 & -594 & 1046 & -1679 & 2548 & -3575 & 4795 & -6035 & 7245 & -8187 & 8874 & -9074 & 8874 & -8187 & 7245 & -6035 & 4795 & -3575 & 2548 & -1679 & 1046 & -594 & 315 & -144 & 59 & -18 & 4 \\
-6 & 28 & -92 & 224 & -486 & 923 & -1624 & 2593 & -3902 & 5473 & -7290 & 9111 & -10884 & 12287 & -13258 & 13545 & -13258 & 12287 & -10884 & 9111 & -7290 & 5473 & -3902 & 2593 & -1624 & 923 & -486 & 224 & -92 & 28 & -6 \\
4 & -18 & 62 & -152 & 333 & -632 & 1117 & -1784 & 2691 & -3770 & 5036 & -6301 & 7539 & -8514 & 9205 & -9398 & 9205 & -8514 & 7539 & -6301 & 5036 & -3770 & 2691 & -1784 & 1117 & -632 & 333 & -152 & 62 & -18 & 4 \\
-1 & 4 & -17 & 42 & -96 & 178 & -314 & 491 & -741 & 1008 & -1330 & 1641 & -1960 & 2166 & -2351 & 2395 & -2351 & 2166 & -1960 & 1641 & -1330 & 1008 & -741 & 491 & -314 & 178 & -96 & 42 & -17 & 4 & -1 \\
0 & 0 & 2 & -5 & 13 & -19 & 29 & -27 & 22 & 10 & -46 & 121 & -179 & 246 & -283 & 322 & -283 & 246 & -179 & 121 & -46 & 10 & 22 & -27 & 29 & -19 & 13 & -5 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 4 & -12 & 22 & -44 & 66 & -98 & 122 & -153 & 165 & -176 & 165 & -153 & 122 & -98 & 66 & -44 & 22 & -12 & 4 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -3 & 5 & -8 & 13 & -14 & 18 & -19 & 21 & -19 & 18 & -14 & 13 & -8 & 5 & -3 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -3 & 4 & -5 & 5 & -5 & 4 & -3 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$H^{10_{125}}_{[21]} = \frac{1}{q^{32}A^{6}} \cdot \tag{138}$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -7 & 7 & -21 & 28 & -49 & 42 & -70 & 63 & -84 & 42 & -70 & 49 & -70 & 42 & -84 & 63 & -70 & 42 & -49 & 28 & -21 & 7 & -7 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 7 & 0 & 14 & 0 & 28 & 28 & 21 & 70 & 0 & 133 & 7 & 154 & 21 & 154 & 49 & 154 & 21 & 154 & 7 & 133 & 0 & 70 & 21 & 28 & 28 & 0 & 14 & 0 & 7 & 0 & 0 \\
-7 & 7 & -28 & 21 & -84 & 35 & -140 & 28 & -203 & -21 & -245 & -84 & -301 & -126 & -343 & -147 & -378 & -147 & -343 & -126 & -301 & -84 & -245 & -21 & -203 & 28 & -140 & 35 & -84 & 21 & -28 & 7 & -7 \\
7 & 0 & 28 & -7 & 84 & -7 & 161 & 7 & 252 & 49 & 336 & 112 & 399 & 189 & 462 & 210 & 483 & 210 & 462 & 189 & 399 & 112 & 336 & 49 & 252 & 7 & 161 & -7 & 84 & -7 & 28 & 0 & 7 \\
0 & -7 & -7 & -21 & -21 & -42 & -63 & -70 & -119 & -105 & -182 & -154 & -224 & -196 & -252 & -231 & -266 & -231 & -252 & -196 & -224 & -154 & -182 & -105 & -119 & -70 & -63 & -42 & -21 & -21 & -7 & -7 & 0 \\
0 & 0 & 0 & 7 & 7 & 21 & 14 & 35 & 35 & 49 & 63 & 56 & 84 & 70 & 98 & 77 & 91 & 77 & 98 & 70 & 84 & 56 & 63 & 49 & 35 & 35 & 14 & 21 & 7 & 7 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -7 & 0 & -14 & 7 & -28 & 7 & -28 & 7 & -28 & 7 & -35 & 7 & -28 & 7 & -28 & 7 & -28 & 7 & -14 & 0 & -7 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

11.3 The first thick knots

Thick are the knots, for which the fundamental superpolynomials are *not* obtained from HOMFLY by a change of variables and Khovanov homologies have non-trivial entries off critical diagonals (marked in red in [21]). In most cases, these superpolynomials have more terms than the HOMFLY polynomial (though this discrepancy can often be eliminated by switching to differential expansion *a la* [15]).

The first thick knots in the Rolfsen table of [21] are

$$8_{19},\ \ 9_{42},\ \ 10_{124},\ 10_{128},\ 10_{132},\ 10_{136},\ 10_{139}, 10_{145},\ 10_{152},\ 10_{153},\ 10_{154},\ 10_{161} \tag{139}$$

The next have eleven and more intersections.

11.3.1 3-strand cases

Five of these are 3-strand and therefore the answers for $H_{[21]}$ are easily available by the methods of [16]:

$$H^{8_{19}}_{[21]} = \frac{1}{q^{30}A^{30}}\cdot \tag{140}$$

$$\begin{pmatrix}
1 & 0 & 2 & 1 & 3 & 1 & 5 & 3 & 6 & 3 & 8 & 4 & 10 & 3 & 10 & 5 & 10 & 3 & 10 & 4 & 8 & 3 & 6 & 3 & 5 & 1 & 3 & 1 & 2 & 0 & 1 \\
-1 & -1 & -3 & -3 & -6 & -6 & -10 & -10 & -16 & -14 & -21 & -16 & -25 & -19 & -27 & -19 & -27 & -19 & -25 & -16 & -21 & -14 & -16 & -10 & -10 & -6 & -6 & -3 & -3 & -1 & -1 \\
0 & 1 & 1 & 3 & 4 & 8 & 9 & 13 & 15 & 20 & 22 & 26 & 27 & 31 & 29 & 32 & 29 & 31 & 27 & 26 & 22 & 20 & 15 & 13 & 9 & 8 & 4 & 3 & 1 & 1 & 0 \\
0 & 0 & 0 & -1 & -1 & -3 & -4 & -7 & -8 & -13 & -13 & -18 & -17 & -22 & -18 & -25 & -18 & -22 & -17 & -18 & -13 & -13 & -8 & -7 & -4 & -3 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 3 & 3 & 5 & 5 & 7 & 8 & 8 & 8 & 8 & 8 & 7 & 5 & 5 & 3 & 3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -2 & -1 & -2 & -1 & -2 & -1 & -2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$H^{10_{124}}_{[21]} = \frac{1}{q^{40}A^{36}}\cdot \tag{141}$$

$$\begin{pmatrix}
1 & 0 & 2 & 1 & 3 & 3 & 4 & 5 & 7 & 7 & 8 & 10 & 11 & 13 & 12 & 15 & 13 & 18 & 13 & 18 & 15 & 18 & 13 & 18 & 13 & 15 & 12 & 13 & 11 & 10 & 8 & 7 & 7 & 5 & 4 & 3 & 3 & 1 & 2 & 0 & 1 \\
-1 & -1 & -3 & -4 & -6 & -9 & -11 & -16 & -18 & -25 & -27 & -35 & -35 & -45 & -43 & -55 & -49 & -61 & -52 & -65 & -54 & -65 & -52 & -61 & -49 & -55 & -43 & -45 & -35 & -35 & -27 & -25 & -18 & -16 & -11 & -9 & -6 & -4 & -3 & -1 & -1 \\
0 & 1 & 1 & 4 & 4 & 10 & 11 & 19 & 21 & 31 & 35 & 46 & 50 & 62 & 65 & 76 & 76 & 87 & 85 & 92 & 86 & 92 & 85 & 87 & 76 & 76 & 65 & 62 & 50 & 46 & 35 & 31 & 21 & 19 & 11 & 10 & 4 & 4 & 1 & 1 & 0 \\
0 & 0 & 0 & -1 & -1 & -4 & -5 & -9 & -13 & -17 & -24 & -29 & -38 & -41 & -53 & -53 & -64 & -63 & -72 & -67 & -76 & -67 & -72 & -63 & -64 & -53 & -53 & -41 & -38 & -29 & -24 & -17 & -13 & -9 & -5 & -4 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 4 & 4 & 9 & 8 & 15 & 14 & 23 & 21 & 29 & 26 & 33 & 29 & 34 & 29 & 33 & 26 & 29 & 21 & 23 & 14 & 15 & 8 & 9 & 4 & 4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -3 & -3 & -4 & -5 & -5 & -8 & -6 & -9 & -6 & -9 & -6 & -8 & -5 & -5 & -4 & -3 & -3 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & -1 & 2 & 0 & 2 & -1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \tag{142}$$

$$H^{10_{139}}_{[21]} = \frac{A^{12}}{q^{22}}\cdot \tag{143}$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -2 & 2 & -3 & 2 & -3 & 2 & -2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 3 & -2 & 3 & -2 & 5 & -3 & 3 & -3 & 5 & -2 & 3 & -2 & 3 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & 2 & -5 & 4 & -9 & 10 & -13 & 12 & -16 & 12 & -13 & 10 & -9 & 4 & -5 & 2 & -1 & 0 & -1 & 0 \\
0 & -1 & 2 & -1 & 2 & -1 & 1 & 3 & 0 & 4 & -1 & 5 & -1 & 4 & 0 & 3 & 1 & -1 & 2 & -1 & 2 & -1 & 0 \\
-1 & 2 & -3 & 4 & -8 & 8 & -9 & 7 & -10 & 5 & -8 & 1 & -8 & 5 & -10 & 7 & -9 & 8 & -8 & 4 & -3 & 2 & -1 \\
1 & -1 & 2 & -3 & 3 & -1 & 1 & 4 & -6 & 12 & -11 & 16 & -11 & 12 & -6 & 4 & 1 & -1 & 3 & -3 & 2 & -1 & 1 \\
0 & 0 & -1 & 0 & 1 & -4 & 7 & -12 & 16 & -24 & 24 & -26 & 24 & -24 & 16 & -12 & 7 & -4 & 1 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & -1 & 4 & -6 & 9 & -11 & 14 & -11 & 9 & -6 & 4 & -1 & 1 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}$$

$$H_{[21]}^{10_{152}} = \frac{A^{24}}{q^{40}}\cdot \tag{144}$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & -4 & 12 & -8 & 22 & -14 & 22 & -14 & 22 & -8 & 12 & -4 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -10 & -6 & -18 & -18 & -14 & -46 & -6 & -74 & 10 & -92 & 8 & -92 & 10 & -74 & -6 & -46 & -14 & -18 & -18 & -6 & -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & 8 & 16 & 26 & 36 & 42 & 74 & 66 & 114 & 94 & 134 & 138 & 130 & 172 & 124 & 172 & 130 & 138 & 134 & 94 & 114 & 66 & 74 & 42 & 36 & 26 & 16 & 8 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -4 & -2 & -18 & -16 & -38 & -50 & -66 & -108 & -92 & -202 & -94 & -324 & -76 & -444 & -54 & -530 & -40 & -564 & -40 & -530 & -54 & -444 & -76 & -324 & -94 & -202 & -92 & -108 & -66 & -50 & -38 & -16 & -18 & -2 & -4 & 0 & 0 & 0 \\
0 & 4 & 0 & 14 & 12 & 34 & 30 & 72 & 58 & 128 & 96 & 190 & 166 & 214 & 276 & 196 & 416 & 146 & 532 & 104 & 576 & 104 & 532 & 146 & 416 & 196 & 276 & 214 & 166 & 190 & 96 & 128 & 58 & 72 & 30 & 34 & 12 & 14 & 0 & 4 & 0 \\
-2 & -4 & -4 & -16 & -10 & -32 & -22 & -64 & -28 & -124 & -14 & -206 & 8 & -284 & -4 & -316 & -42 & -308 & -96 & -294 & -116 & -294 & -96 & -308 & -42 & -316 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & -64 & -22 & -32 & -10 & -16 & -4 & -4 & -2 \\
2 & 0 & 4 & 6 & 0 & 16 & 4 & 22 & 6 & 36 & 0 & 70 & -28 & 116 & -58 & 152 & -70 & 168 & -68 & 164 & -60 & 164 & -68 & 168 & -70 & 152 & -58 & 116 & -28 & 70 & 0 & 36 & 6 & 22 & 4 & 16 & 0 & 6 & 4 & 0 & 2
\end{pmatrix}$$

$$H_{[21]}^{10_{161}} = \frac{A^{18}}{q^{30}}\cdot \tag{145}$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 4 & -4 & 8 & -16 & 16 & -14 & 16 & -16 & 8 & -4 & 4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -2 & 2 & -2 & 4 & -8 & 8 & -8 & 10 & -8 & 8 & -8 & 4 & -2 & 2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 4 & -4 & 8 & -10 & 16 & -14 & 12 & -12 & 8 & -4 & 0 & -4 & 8 & -12 & 12 & -14 & 16 & -10 & 8 & -4 & 4 & 0 & 2 & 0 & 0 & 0 \\
0 & -2 & 2 & -4 & 6 & -6 & 4 & -2 & 4 & 2 & -2 & 6 & -4 & 14 & -8 & 14 & -8 & 14 & -4 & 6 & -2 & 2 & 4 & -2 & 4 & -6 & 6 & -4 & 2 & -2 & 0 \\
-2 & 4 & -4 & 2 & -4 & 2 & 2 & -10 & 10 & -12 & 10 & -16 & 6 & -2 & -2 & -4 & -2 & -2 & 6 & -16 & 10 & -12 & 10 & -10 & 2 & 2 & -4 & 2 & -4 & 4 & -2 \\
0 & -2 & 2 & -4 & 0 & -2 & -2 & 2 & -4 & -6 & 8 & -22 & 28 & -44 & 44 & -50 & 44 & -44 & 28 & -22 & 8 & -6 & -4 & 2 & -2 & -2 & 0 & -4 & 2 & -2 & 0 \\
2 & 0 & 0 & 4 & -2 & 2 & 2 & 4 & 2 & 0 & -4 & 20 & -22 & 32 & -38 & 50 & -38 & 32 & -22 & 20 & -4 & 0 & 2 & 4 & 2 & 2 & -2 & 4 & 0 & 0 & 2
\end{pmatrix}$$

Moreover, two of these are torus knots, $8_{19} = Torus[3,4]$ and $10_{124} = Torus[3,5]$, and therefore *arbitrary* colored HOMFLY polynomials for them are available: provided by the Rosso-Jones formula [32, 6].

11.3.2 4-parallel pretzel finger cases

Three of the above thick knots are of the pretzel type and are described by the simple formulas:

$$H_R^{Pr(n_1,n_2,\overline{n_3})} = \overline{\sum_{X\in R\otimes\bar R}} \underbrace{\frac{d_R^2}{\sqrt{\bar d_X}}}_{\frac{d_R}{\bar S_{1X}}} \bar{\mathcal{A}}^{\rm par}_{1X}(n_1)\bar{\mathcal{A}}^{\rm par}_{1X}(n_2)\bar{\mathcal{A}}^{\rm ea}_{1X}(\overline{n_3}) \tag{146}$$

and

$$H_R^{Pr(n_1,n_2,n_3,n_4)} = \overline{\sum_{X\in R\otimes\bar R}} \underbrace{\frac{d_R^3}{\bar d_X}}_{\frac{d_R}{\bar S^2_{1X}}} \bar{\mathcal{A}}^{\rm par}_{1X}(n_1)\bar{\mathcal{A}}^{\rm par}_{1X}(n_2)\bar{\mathcal{A}}^{\rm par}_{1X}(n_3)\bar{\mathcal{A}}^{\rm par}_{1X}(n_4) \tag{147}$$

These three cases are 3-strand (and thus already known)

$$8_{19} = Torus[3,4] = Pr(3,3,-\bar 2) \quad\Longrightarrow\quad H^{8_{19}} = (140) \tag{148}$$

$$10_{124} = Torus[3,5] = Pr(5,3,-\bar 2) = Pr(2,-1,5,3) \quad\Longrightarrow\quad H^{10_{124}} = (142) \tag{149}$$

and

$$10_{139} = Pr(4,-1,3,3) \quad\Longrightarrow\quad H^{10_{139}} = (144) \tag{150}$$

11.3.3 Realizations from [19]

According to [18] and [19], seven thick knots from [4] can be realized just as triple-finger starfish diagrams:

$$H_R^{8_{19}} = \overline{\sum_{X\in R\otimes\bar R}} \frac{d_R}{\bar S_{1X}} (ST^3S^\dagger)_{1X}(ST^3S^\dagger)_{1X}(\bar S\bar T^{-2}\bar S)_{1X} \quad\Longrightarrow\quad H_R^{8_{19}} = (140) \tag{151}$$

$$H_R^{9_{42}} = \overline{\sum_{X\in R\otimes\bar R}} \frac{d_R}{\bar S_{1X}} (ST^3S^\dagger)_{1X}(ST^{-2}S^\dagger)_{1X}(STS^\dagger\bar T^{-1}\bar S\bar T^2\bar S)_{1X} \tag{152}$$

$$H_{[21]}^{9_{42}} = \frac{1}{q^{22}A^8}\cdot \tag{153}$$

$$\begin{pmatrix}
0&0&0&0&0&0&1&0&2&-1&2&0&2&-1&2&0&1&0&0&0&0&0&0\\
0&0&0&-1&-1&-2&-1&-2&-3&-3&-4&-2&-4&-3&-3&-2&-1&-2&-1&-1&0&0&0\\
0&1&1&1&3&2&6&2&8&4&9&4&9&4&8&2&6&2&3&1&1&1&0\\
-1&1&-5&3&-9&4&-15&5&-19&5&-21&5&-21&5&-19&5&-15&4&-9&3&-5&1&-1\\
1&-2&4&-3&9&-6&14&-7&20&-10&24&-10&24&-10&20&-7&14&-6&9&-3&4&-2&1\\
0&0&0&0&-1&0&-4&1&-8&4&-12&4&-12&4&-8&1&-4&0&-1&0&0&0&0\\
0&0&0&0&-1&1&-1&0&3&-1&5&-4&5&-1&3&0&-1&1&-1&0&0&0&0\\
0&0&0&0&0&0&0&1&-1&2&-3&2&-3&2&-1&1&0&0&0&0&0&0&0
\end{pmatrix}$$

$$H_R^{10_{124}} = \overline{\sum_{X\in R\otimes\bar R}} \frac{d_R}{\bar S_{1X}} (ST^5S^\dagger)_{1X}(ST^2S^\dagger)_{1X}(ST^2S^\dagger\bar T^{-1}\bar S)_{1X} \quad\Longrightarrow\quad H_R^{10_{124}} = (142) \tag{154}$$

$$H_R^{10_{128}} = \overline{\sum_{X\in R\otimes\bar{R}} \frac{d_R}{\bar{S}_{1X}}} (\bar{S}\bar{T}^{-2}\bar{S})_{1X} (\bar{S}\bar{T}^{-1}ST^2S^\dagger)_{1X} (\bar{S}\bar{T}^{-3}ST^2S^\dagger)_{1X} \tag{155}$$

$$H_{[21]}^{10_{128}} = \frac{1}{q^{34}A^{36}} \cdot \tag{156}$$

$$\begin{pmatrix}
0 & 0 & 1 & -2 & 3 & -4 & 7 & -10 & 12 & -12 & 15 & -17 & 17 & -16 & 19 & -19 & 19 & -18 & 19 & -19 & 19 & -16 & 17 & -17 & 15 & -12 & 12 & -10 & 7 & -4 & 3 & -2 & 1 & 0 & 0 \\
0 & 1 & -2 & 5 & -8 & 14 & -19 & 25 & -28 & 35 & -36 & 38 & -39 & 44 & -41 & 43 & -43 & 46 & -43 & 43 & -41 & 44 & -39 & 38 & -36 & 35 & -28 & 25 & -19 & 14 & -8 & 5 & -2 & 1 & 0 \\
1 & -2 & 4 & -6 & 10 & -15 & 18 & -21 & 25 & -29 & 27 & -32 & 31 & -33 & 31 & -35 & 31 & -34 & 31 & -35 & 31 & -33 & 31 & -32 & 27 & -29 & 25 & -21 & 18 & -15 & 10 & -6 & 4 & -2 & 1 \\
-1 & 0 & -2 & 0 & -4 & 0 & -4 & 0 & -4 & -3 & -3 & -5 & 2 & -8 & 5 & -12 & 7 & -12 & 7 & -12 & 5 & -8 & 2 & -5 & -3 & -3 & -4 & 0 & -4 & 0 & -4 & 0 & -2 & 0 & -1 \\
0 & 1 & -1 & 3 & -2 & 7 & -4 & 11 & -8 & 16 & -13 & 22 & -19 & 27 & -25 & 34 & -30 & 34 & -30 & 34 & -25 & 27 & -19 & 22 & -13 & 16 & -8 & 11 & -4 & 7 & -2 & 3 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 2 & -2 & 4 & -3 & 7 & -3 & 9 & -6 & 15 & -6 & 18 & -10 & 18 & -6 & 15 & -6 & 9 & -3 & 7 & -3 & 4 & -2 & 2 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -3 & 0 & -5 & 1 & -9 & -1 & -12 & -1 & -14 & 0 & -16 & 0 & -14 & -1 & -12 & -1 & -9 & 1 & -5 & 0 & -3 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 1 & 5 & 0 & 9 & 0 & 10 & 0 & 9 & 0 & 5 & 1 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -2 & 0 & -2 & 0 & -2 & 0 & -2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$H_R^{10_{132}} = \overline{\sum_{X\in R\otimes\bar{R}} \frac{d_R}{\bar{S}_{1X}}} (ST^{-2}S^\dagger\bar{T}^3\bar{S})_{1X} (ST^{-2}S^\dagger)_{1X} (\bar{S}\bar{T}^{-1}ST^2S^\dagger)_{1X} \tag{157}$$

$$H_{[21]}^{10_{132}} = \frac{A^4}{q^{20}} \cdot \tag{158}$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -1 & 0 & -2 & 1 & -2 & 0 & -2 & 1 & -2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 2 & 1 & 2 & 3 & 3 & 4 & 2 & 4 & 3 & 3 & 2 & 1 & 2 & 1 & 1 & 0 & 0 \\
-1 & 0 & -2 & 1 & -5 & 1 & -8 & 1 & -10 & 1 & -10 & 1 & -10 & 1 & -8 & 1 & -5 & 1 & -2 & 0 & -1 \\
0 & 0 & 2 & -3 & 5 & -3 & 7 & -3 & 5 & 0 & 7 & 0 & 5 & -3 & 7 & -3 & 5 & -3 & 2 & 0 & 0 \\
0 & 1 & -1 & 1 & -1 & 2 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 2 & -1 & 1 & -1 & 1 & 0 \\
1 & -1 & 1 & -2 & 4 & -5 & 6 & -8 & 10 & -11 & 10 & -11 & 10 & -8 & 6 & -5 & 4 & -2 & 1 & -1 & 1 \\
0 & 0 & -1 & 2 & -4 & 6 & -9 & 13 & -15 & 17 & -18 & 17 & -15 & 13 & -9 & 6 & -4 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 3 & -5 & 7 & -9 & 10 & -9 & 7 & -5 & 3 & -1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$H_R^{10_{136}} = \overline{\sum_{X\in R\otimes\bar{R}} \frac{d_R}{\bar{S}_{1X}}} (\bar{S}\bar{T}^{-2}\bar{S}\bar{T}^2\bar{S})_{1X} (ST^2S^\dagger)_{1X} (\bar{S}\bar{T}^{-3}ST^{-2}S^\dagger)_{1X} \tag{159}$$

$$H_{[21]}^{10_{136}} = \frac{1}{q^{22}A^{12}} \cdot \tag{160}$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 3 & -4 & 5 & -5 & 5 & -4 & 3 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & -2 & 3 & -4 & 3 & -5 & 4 & -4 & 4 & -5 & 3 & -4 & 3 & -2 & 1 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 4 & -4 & 11 & -17 & 24 & -24 & 33 & -33 & 33 & -24 & 24 & -17 & 11 & -4 & 4 & -1 & 0 & 1 & 0 \\
-2 & 5 & -14 & 21 & -34 & 46 & -67 & 74 & -92 & 98 & -110 & 103 & -110 & 98 & -92 & 74 & -67 & 46 & -34 & 21 & -14 & 5 & -2 \\
4 & -9 & 17 & -21 & 37 & -43 & 59 & -62 & 84 & -82 & 99 & -91 & 99 & -82 & 84 & -62 & 59 & -43 & 37 & -21 & 17 & -9 & 4 \\
-2 & 2 & -3 & 0 & -6 & -1 & -7 & 1 & -16 & 10 & -26 & 12 & -26 & 10 & -16 & 1 & -7 & -1 & -6 & 0 & -3 & 2 & -2 \\
0 & 1 & 0 & 3 & -2 & 5 & 3 & 2 & 14 & -10 & 24 & -14 & 24 & -10 & 14 & 2 & 3 & 5 & -2 & 3 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & -2 & -5 & 10 & -21 & 23 & -38 & 35 & -38 & 23 & -21 & 10 & -5 & -2 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 & 5 & -5 & 8 & -7 & 8 & -5 & 5 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$H_R^{10_{145}} = \sum_{X \in R\otimes R} \frac{d_R}{S_{1X}} (ST^{-2}S^\dagger \bar{T}^2 S)_{1X} (\bar{S}\bar{T}^3 S)_{1X} (ST^{-1}S^\dagger \bar{T}\bar{S}\bar{T}^{-1}S)_{1X} \tag{161}$$

10_{145} is the only example in our thick-knot set, where summation is over representations $X \in R\otimes R$, i.e. in the *parallel* sector. *This* summation is *insensitive* to the choice of parameters ξ in sec.7, thus there is no overline in this case.

$$H_{[21]}^{10_{145}} = \frac{A^{12}}{q^{22}}\cdot \tag{162}$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -2 & 2 & -3 & 2 & -3 & 2 & -2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 3 & -2 & 3 & -2 & 5 & -3 & 3 & -3 & 5 & -2 & 3 & -2 & 3 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & 2 & -5 & 4 & -9 & 10 & -13 & 12 & -16 & 12 & -13 & 10 & -9 & 4 & -5 & 2 & -1 & 0 & -1 & 0 \\
0 & -1 & 2 & -1 & 2 & -1 & 1 & 3 & 0 & 4 & -1 & 5 & -1 & 4 & 0 & 3 & 1 & -1 & 2 & -1 & 2 & -1 & 0 \\
-1 & 2 & -3 & 4 & -8 & 8 & -9 & 7 & -10 & 5 & -8 & 1 & -8 & 5 & -10 & 7 & -9 & 8 & -8 & 4 & -3 & 2 & -1 \\
1 & -1 & 2 & -3 & 3 & -1 & 1 & 4 & -6 & 12 & -11 & 16 & -11 & 12 & -6 & 4 & 1 & -1 & 3 & -3 & 2 & -1 & 1 \\
0 & 0 & -1 & 0 & 1 & -4 & 7 & -12 & 16 & -24 & 24 & -26 & 24 & -24 & 16 & -12 & 7 & -4 & 1 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & -1 & 4 & -6 & 9 & -11 & 14 & -11 & 9 & -6 & 4 & -1 & 1 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}$$

Three knots are more complicated, their HOMFLY polynomials are represented by double sums, at best:

$$H_R^{10_{152}} = \overline{\sum_{X,Y\in R\otimes\bar{R}}} \frac{d_R}{\bar{S}_{1X}\bar{S}_{1Y}} (\bar{S}\bar{T}^2\bar{S})_{1X} (ST^{-3}S^\dagger)_{1X} \left(\bar{T}\bar{S}\right)_{YX} (ST^{-3}S^\dagger)_{1Y} (\bar{S}\bar{T}ST^{-1}S^\dagger)_{1Y} \tag{163}$$

$$H_{[21]}^{10_{152}} = \frac{A^{24}}{q^{40}}\cdot \tag{164}$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -2 & 6 & -4 & 11 & -7 & 11 & -7 & 11 & -4 & 6 & -2 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & -3 & -9 & -9 & -7 & -23 & -3 & -37 & 5 & -46 & 4 & -46 & 5 & -37 & -3 & -23 & -7 & -9 & -9 & -3 & -5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 & 8 & 13 & 18 & 21 & 37 & 33 & 57 & 47 & 67 & 69 & 65 & 86 & 62 & 86 & 65 & 69 & 67 & 47 & 57 & 33 & 37 & 21 & 18 & 13 & 8 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & -1 & -9 & -8 & -19 & -25 & -33 & -54 & -46 & -101 & -47 & -162 & -38 & -222 & -27 & -265 & -20 & -282 & -20 & -265 & -27 & -222 & -38 & -162 & -47 & -101 & -46 & -54 & -33 & -25 & -19 & -8 & -9 & -1 & -2 & 0 & 0 & 0 \\
0 & 2 & 0 & 7 & 6 & 17 & 15 & 36 & 29 & 64 & 48 & 95 & 83 & 107 & 138 & 98 & 208 & 73 & 266 & 52 & 288 & 52 & 266 & 73 & 208 & 98 & 138 & 107 & 83 & 95 & 48 & 64 & 29 & 36 & 15 & 17 & 6 & 7 & 0 & 2 & 0 \\
-1 & -2 & -2 & -8 & -5 & -16 & -11 & -32 & -14 & -62 & -7 & -103 & 4 & -142 & -2 & -158 & -21 & -154 & -48 & -147 & -58 & -147 & -48 & -154 & -21 & -158 & -2 & -142 & 4 & -103 & -7 & -62 & -14 & -32 & -11 & -16 & -5 & -8 & -2 & -2 & -1 \\
1 & 0 & 2 & 3 & 0 & 8 & 2 & 11 & 3 & 18 & 0 & 35 & -14 & 58 & -29 & 76 & -35 & 84 & -34 & 82 & -30 & 82 & -34 & 84 & -35 & 76 & -29 & 58 & -14 & 35 & 0 & 18 & 3 & 11 & 2 & 8 & 0 & 3 & 2 & 0 & 1
\end{pmatrix}$$

$$H_R^{10_{153}} = \overline{\sum_{X,Y\in R\otimes\bar{R}}} \frac{d_R}{\bar{S}_{1X}\bar{S}_{1Y}} (ST^3S^\dagger)_{1X} (ST^2S^\dagger)_{1X} \bar{S}_{XY} (ST^{-2}S^\dagger\bar{T}\bar{S})_{1Y} (ST^{-1}S^\dagger\bar{T}\bar{S})_{1Y} \tag{165}$$

$$H_{[21]}^{10_{153}} = \frac{1}{q^{32}A^6}\cdot \tag{166}$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -3 & 4 & -5 & 5 & -5 & 4 & -3 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -5 & 8 & -11 & 13 & -15 & 13 & -11 & 8 & -5 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -3 & 5 & -8 & 13 & -12 & 15 & -14 & 19 & -14 & 15 & -12 & 13 & -8 & 5 & -3 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 3 & -3 & 6 & -3 & 6 & -1 & 2 & 2 & 2 & -3 & 5 & -8 & 10 & -8 & 5 & -3 & 2 & 2 & 2 & -1 & 6 & -3 & 6 & -3 & 3 & -1 & 1 & 0 & 0 \\
-1 & 1 & -4 & 4 & -11 & 8 & -17 & 12 & -23 & 14 & -26 & 20 & -31 & 23 & -35 & 29 & -34 & 29 & -35 & 23 & -31 & 20 & -26 & 14 & -23 & 12 & -17 & 8 & -11 & 4 & -4 & 1 & -1 \\
1 & 0 & 2 & -1 & 6 & -4 & 11 & -11 & 17 & -15 & 15 & -15 & 10 & -14 & 3 & -12 & 5 & -12 & 3 & -14 & 10 & -15 & 15 & -15 & 17 & -11 & 11 & -4 & 6 & -1 & 2 & 0 & 1 \\
0 & 0 & 1 & -1 & 3 & 1 & 5 & 5 & 6 & 11 & 15 & 9 & 26 & 4 & 43 & -4 & 49 & -4 & 43 & 4 & 26 & 9 & 15 & 11 & 6 & 5 & 5 & 1 & 3 & -1 & 1 & 0 & 0 \\
0 & -1 & 0 & -2 & -2 & -4 & -6 & -6 & -12 & -11 & -16 & -15 & -26 & -12 & -33 & -8 & -43 & -8 & -33 & -12 & -26 & -15 & -16 & -11 & -12 & -6 & -6 & -4 & -2 & -2 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 1 & 2 & 2 & 4 & 6 & 4 & 9 & 5 & 13 & 7 & 13 & 8 & 12 & 8 & 13 & 7 & 13 & 5 & 9 & 4 & 6 & 4 & 2 & 2 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -2 & 1 & -4 & 1 & -4 & 1 & -4 & 1 & -5 & 1 & -4 & 1 & -4 & 1 & -4 & 1 & -2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$H_R^{10_{154}} = \overline{\sum_{X,Y\in R\otimes\bar R}} \frac{d_R}{\bar S_{1X}\bar S_{1Y}} (\bar S\bar T S T^{-1} S^\dagger \bar T\bar S)_{1X} (\bar S\bar T^2\bar S)_{1X} \bar S_{XY} (ST^{-2}S^\dagger\bar T\bar S)_{1Y} (ST^{-1}S^\dagger\bar T\bar S)_{1Y} \tag{167}$$

$$H_{[21]}^{10_{154}} = \frac{1}{q^{30}A^{36}}\cdot \tag{168}$$

$$\begin{pmatrix}
1 & 0 & 0 & 4 & -2 & 2 & 5 & -6 & 20 & -24 & 33 & -29 & 35 & -24 & 26 & -18 & 26 & -24 & 35 & -29 & 33 & -24 & 20 & -6 & 5 & 2 & -2 & 4 & 0 & 0 & 1\\
0 & -2 & 2 & -3 & -5 & 8 & -23 & 29 & -48 & 48 & -59 & 44 & -47 & 34 & -35 & 18 & -35 & 34 & -47 & 44 & -59 & 48 & -48 & 29 & -23 & 8 & -5 & -3 & 2 & -2 & 0\\
-2 & 3 & -4 & 2 & -2 & 0 & 4 & -10 & 9 & -10 & 4 & -7 & -5 & 8 & -17 & 6 & -17 & 8 & -5 & -7 & 4 & -10 & 9 & -10 & 4 & 0 & -2 & 2 & -4 & 3 & -2\\
1 & -2 & 3 & -5 & 12 & -12 & 16 & -13 & 20 & -8 & 13 & 9 & 2 & 24 & -8 & 32 & -8 & 24 & 2 & 9 & 13 & -8 & 20 & -13 & 16 & -12 & 12 & -5 & 3 & -2 & 1\\
0 & 1 & -1 & 3 & -1 & 4 & -1 & 1 & 3 & -9 & 21 & -41 & 52 & -79 & 84 & -98 & 84 & -79 & 52 & -41 & 21 & -9 & 3 & 1 & -1 & 4 & -1 & 3 & -1 & 1 & 0\\
0 & 0 & 0 & -1 & -2 & -2 & -2 & -2 & -8 & 5 & -17 & 21 & -37 & 37 & -47 & 50 & -47 & 37 & -37 & 21 & -17 & 5 & -8 & -2 & -2 & -2 & -2 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 4 & -2 & 5 & 2 & 0 & -2 & 0 & 8 & 0 & -2 & 0 & 2 & 5 & -2 & 4 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 2 & -1 & 6 & -1 & 2 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & -2 & 2 & -2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

One can also represent in the same form the simplest thick knot 9_{42} :

$$H_R^{9_{42}} = \overline{\sum_{X,Y\in R\otimes\bar R}} \frac{d_R}{\bar S_{1X}\bar S_{1Y}} (ST^3S^\dagger)_{1X} (ST^{-2}S^\dagger)_{1X} \left(\bar S\bar T^2\bar S\right)_{XY} (STS^\dagger)_{1Y} (STS^\dagger)_{1Y} = (152) \tag{169}$$

Note that in formulas of this section we do not make any difference between the products $T_+^mT_-^n$ and T_-^{m+n}, see (6). This is because such difference gets observable only for mutants, which appear only at 11 intersections (alternatively we can claim that all these knots can be represented by diagrams without combinations $T_+^mT_-^n$ with both $m,n\neq 0$).

11.4 Mutant pairs

$$H_{[21]}^{11n73} = \frac{1}{q^{34}A^{12}}\cdot \tag{170}$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -4 & 2 & -8 & 4 & -13 & 6 & -16 & 8 & -20 & 8 & -16 & 6 & -13 & 4 & -8 & 2 & -4 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 2 & 4 & 7 & 5 & 20 & 2 & 46 & -8 & 75 & -19 & 103 & -28 & 108 & -28 & 103 & -19 & 75 & -8 & 46 & 2 & 20 & 5 & 7 & 4 & 2 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & -2 & -1 & -8 & -6 & -20 & -14 & -44 & -31 & -80 & -48 & -120 & -81 & -139 & -109 & -148 & -134 & -148 & -109 & -139 & -81 & -120 & -48 & -80 & -31 & -44 & -14 & -20 & -6 & -8 & -1 & -2 & 0 & 0\\
1 & 0 & 7 & 3 & 19 & 8 & 52 & 17 & 115 & 21 & 203 & 44 & 279 & 91 & 326 & 161 & 329 & 187 & 329 & 161 & 326 & 91 & 279 & 44 & 203 & 21 & 115 & 17 & 52 & 8 & 19 & 3 & 7 & 0 & 1\\
-3 & 0 & -9 & -6 & -20 & -27 & -38 & -68 & -67 & -131 & -103 & -221 & -138 & -320 & -162 & -396 & -177 & -428 & -177 & -396 & -162 & -320 & -138 & -221 & -103 & -131 & -67 & -68 & -38 & -27 & -20 & -6 & -9 & 0 & -3\\
3 & 0 & 7 & 8 & 10 & 39 & 1 & 98 & -10 & 183 & -10 & 262 & 18 & 322 & 52 & 359 & 82 & 374 & 82 & 359 & 52 & 322 & 18 & 262 & -10 & 183 & -10 & 98 & 1 & 39 & 10 & 8 & 7 & 0 & 3\\
-1 & 0 & -5 & -3 & -8 & -12 & -14 & -33 & -24 & -51 & -50 & -61 & -100 & -39 & -176 & -14 & -223 & 12 & -223 & -14 & -176 & -39 & -100 & -61 & -50 & -51 & -24 & -33 & -14 & -12 & -8 & -3 & -5 & 0 & -1\\
0 & 0 & 2 & -1 & 7 & -4 & 18 & -9 & 32 & -10 & 45 & -11 & 62 & -12 & 83 & -35 & 107 & -32 & 107 & -35 & 83 & -12 & 62 & -11 & 45 & -10 & 32 & -9 & 18 & -4 & 7 & -1 & 2 & 0 & 0\\
0 & 0 & 0 & 0 & -1 & 0 & -3 & 2 & -5 & -2 & 0 & -17 & 23 & -50 & 59 & -91 & 90 & -106 & 90 & -91 & 59 & -50 & 23 & -17 & 0 & -2 & -5 & 2 & -3 & 0 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 5 & -9 & 15 & -20 & 27 & -32 & 38 & -40 & 42 & -40 & 38 & -32 & 27 & -20 & 15 & -9 & 5 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$H_{[21]}^{11n74} = \frac{1}{q^{36}A^{12}} \cdot \tag{171}$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -4 & 2 & -8 & 4 & -13 & 6 & -16 & 8 & -20 & 8 & -16 & 6 & -13 & 4 & -8 & 2 & -4 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 8 & 2 & 25 & -7 & 61 & -29 & 103 & -54 & 143 & -72 & 154 & -72 & 143 & -54 & 103 & -29 & 61 & -7 & 25 & 2 & 8 & 4 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & -1 & -10 & -1 & -27 & -2 & -62 & -10 & -105 & -20 & -145 & -58 & -161 & -93 & -161 & -120 & -161 & -93 & -161 & -58 & -145 & -20 & -105 & -10 & -62 & -2 & -27 & -1 & -10 & -1 & -2 & 0 & 0 & 0 \\
0 & 1 & 1 & 6 & 2 & 22 & 0 & 70 & -11 & 154 & -34 & 268 & -26 & 356 & 15 & 396 & 92 & 394 & 127 & 394 & 92 & 396 & 15 & 356 & -26 & 268 & -34 & 154 & -11 & 70 & 0 & 22 & 2 & 6 & 1 & 1 & 0 \\
-1 & 0 & -6 & 1 & -21 & -1 & -49 & -13 & -92 & -44 & -156 & -75 & -255 & -91 & -382 & -85 & -490 & -73 & -534 & -73 & -490 & -85 & -382 & -91 & -255 & -75 & -156 & -44 & -92 & -13 & -49 & -1 & -21 & 1 & -6 & 0 & -1 \\
1 & 0 & 6 & -3 & 23 & -9 & 61 & -24 & 122 & -33 & 208 & -38 & 296 & -29 & 384 & -25 & 453 & -22 & 480 & -22 & 453 & -25 & 384 & -29 & 296 & -38 & 208 & -33 & 122 & -24 & 61 & -9 & 23 & -3 & 6 & 0 & 1 \\
0 & -1 & -1 & -4 & -2 & -11 & -4 & -32 & -5 & -63 & 4 & -115 & 9 & -177 & 37 & -246 & 55 & -288 & 72 & -288 & 55 & -246 & 37 & -177 & 9 & -115 & 4 & -63 & -5 & -32 & -4 & -11 & -2 & -4 & -1 & -1 & 0 \\
0 & 0 & 0 & 2 & -1 & 9 & -9 & 25 & -21 & 50 & -31 & 70 & -39 & 87 & -35 & 105 & -51 & 120 & -46 & 120 & -51 & 105 & -35 & 87 & -39 & 70 & -31 & 50 & -21 & 25 & -9 & 9 & -1 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -3 & 1 & -2 & -7 & 9 & -32 & 44 & -78 & 94 & -131 & 134 & -152 & 134 & -131 & 94 & -78 & 44 & -32 & 9 & -7 & -2 & 1 & -3 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 5 & -9 & 15 & -20 & 27 & -32 & 38 & -40 & 42 & -40 & 38 & -32 & 27 & -20 & 15 & -9 & 5 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$H_{[21]}^{11n76} = \frac{1}{q^{40}A^{24}} \cdot \tag{172}$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -1 & 2 & -7 & 14 & -29 & 47 & -78 & 110 & -156 & 198 & -254 & 297 & -347 & 374 & -404 & 404 & -404 & 374 & -347 & 297 & -254 & 198 & -156 & 110 & -78 & 47 & -29 & 14 & -7 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 6 & -8 & 23 & -32 & 73 & -101 & 186 & -244 & 382 & -456 & 624 & -689 & 858 & -880 & 1018 & -986 & 1076 & -986 & 1018 & -880 & 858 & -689 & 624 & -456 & 382 & -244 & 186 & -101 & 73 & -32 & 23 & -8 & 6 & -1 & 1 & 0 & 0 \\
-1 & 2 & -10 & 18 & -52 & 83 & -176 & 251 & -444 & 567 & -875 & 1012 & -1396 & 1481 & -1904 & 1883 & -2295 & 2160 & -2547 & 2304 & -2630 & 2304 & -2547 & 2160 & -2295 & 1883 & -1904 & 1481 & -1396 & 1012 & -875 & 567 & -444 & 251 & -176 & 83 & -52 & 18 & -10 & 2 & -1 \\
2 & -3 & 16 & -22 & 63 & -75 & 169 & -164 & 321 & -249 & 449 & -219 & 466 & -39 & 335 & 271 & 139 & 573 & -38 & 756 & -89 & 756 & -38 & 573 & 139 & 271 & 335 & -39 & 466 & -219 & 449 & -249 & 321 & -164 & 169 & -75 & 63 & -22 & 16 & -3 & 2 \\
-1 & 0 & -7 & 2 & -21 & -8 & -28 & -69 & 22 & -263 & 193 & -645 & 517 & -1196 & 935 & -1801 & 1335 & -2296 & 1597 & -2563 & 1700 & -2563 & 1597 & -2296 & 1335 & -1801 & 935 & -1196 & 517 & -645 & 193 & -263 & 22 & -69 & -28 & -8 & -21 & 2 & -7 & 0 & -1 \\
0 & 1 & 0 & 4 & 4 & 13 & 11 & 35 & 23 & 92 & 37 & 170 & 67 & 266 & 156 & 302 & 291 & 279 & 451 & 237 & 516 & 237 & 451 & 279 & 291 & 302 & 156 & 266 & 67 & 170 & 37 & 92 & 23 & 35 & 11 & 13 & 4 & 4 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & -4 & -2 & -14 & -12 & -19 & -47 & -11 & -149 & 56 & -314 & 168 & -532 & 284 & -687 & 361 & -762 & 361 & -687 & 284 & -532 & 168 & -314 & 56 & -149 & -11 & -47 & -19 & -12 & -14 & -2 & -4 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 4 & 2 & 11 & 12 & 7 & 45 & -16 & 121 & -82 & 221 & -152 & 299 & -190 & 299 & -152 & 221 & -82 & 121 & -16 & 45 & 7 & 12 & 11 & 2 & 4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -4 & -1 & -8 & -2 & -11 & -4 & -15 & -6 & -16 & -6 & -15 & -4 & -11 & -2 & -8 & -1 & -4 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & -1 & 2 & 0 & 2 & -1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$H_{[21]}^{11n78} = \frac{1}{q^{40}A^{24}} \cdot \tag{173}$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -1 & 2 & -7 & 14 & -29 & 47 & -78 & 110 & -156 & 198 & -254 & 297 & -347 & 374 & -404 & 404 & -404 & 374 & -347 & 297 & -254 & 198 & -156 & 110 & -78 & 47 & -29 & 14 & -7 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 6 & -8 & 23 & -32 & 73 & -101 & 186 & -244 & 382 & -457 & 626 & -691 & 862 & -886 & 1024 & -993 & 1084 & -993 & 1024 & -886 & 862 & -691 & 626 & -457 & 382 & -244 & 186 & -101 & 73 & -32 & 23 & -8 & 6 & -1 & 1 & 0 & 0 \\
-1 & 2 & -10 & 18 & -52 & 83 & -176 & 251 & -444 & 567 & -873 & 1009 & -1394 & 1476 & -1898 & 1880 & -2291 & 2155 & -2547 & 2304 & -2626 & 2304 & -2547 & 2155 & -2291 & 1880 & -1898 & 1476 & -1394 & 1009 & -873 & 567 & -444 & 251 & -176 & 83 & -52 & 18 & -10 & 2 & -1 \\
2 & -3 & 16 & -22 & 63 & -75 & 169 & -165 & 321 & -247 & 447 & -214 & 456 & -29 & 325 & 287 & 127 & 580 & -50 & 765 & -93 & 765 & -50 & 580 & 127 & 287 & 325 & -29 & 456 & -214 & 447 & -247 & 321 & -165 & 169 & -75 & 63 & -22 & 16 & -3 & 2 \\
-1 & 0 & -7 & 2 & -21 & -7 & -30 & -66 & 18 & -258 & 189 & -642 & 513 & -1195 & 933 & -1795 & 1327 & -2286 & 1581 & -2544 & 1684 & -2544 & 1581 & -2286 & 1327 & -1795 & 933 & -1195 & 513 & -642 & 189 & -258 & 18 & -66 & -30 & -7 & -21 & 2 & -7 & 0 & -1 \\
0 & 1 & 0 & 4 & 4 & 12 & 13 & 32 & 27 & 87 & 41 & 167 & 71 & 265 & 158 & 296 & 299 & 269 & 467 & 218 & 532 & 218 & 467 & 269 & 299 & 296 & 158 & 265 & 71 & 167 & 41 & 87 & 27 & 32 & 13 & 12 & 4 & 4 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & -4 & -2 & -13 & -12 & -21 & -45 & -16 & -139 & 46 & -304 & 152 & -520 & 277 & -675 & 352 & -758 & 352 & -675 & 277 & -520 & 152 & -304 & 46 & -139 & -16 & -45 & -21 & -12 & -13 & -2 & -4 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 4 & 2 & 9 & 15 & 5 & 50 & -22 & 124 & -86 & 226 & -152 & 299 & -194 & 299 & -152 & 226 & -86 & 124 & -22 & 50 & 5 & 15 & 9 & 2 & 4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -4 & 0 & -10 & 0 & -15 & 2 & -21 & 1 & -24 & 1 & -21 & 2 & -15 & 0 & -10 & 0 & -4 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & -1 & 2 & 0 & 2 & -1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$H_{[21]}^{11a57} = \frac{1}{q^{40}A^{12}} \cdot \tag{174}$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -4 & 2 & -8 & 4 & -13 & 6 & -16 & 8 & -20 & 8 & -16 & 6 & -13 & 4 & -8 & 2 & -4 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 9 & 0 & 19 & -1 & 38 & 1 & 57 & -2 & 77 & 3 & 86 & -4 & 86 & 3 & 77 & -2 & 57 & 1 & 38 & -1 & 19 & 0 & 9 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -3 & -3 & -16 & 6 & -56 & 35 & -148 & 99 & -292 & 203 & -479 & 320 & -651 & 419 & -762 & 450 & -762 & 419 & -651 & 320 & -479 & 203 & -292 & 99 & -148 & 35 & -56 & 6 & -16 & -3 & -3 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & -1 & 13 & -8 & 40 & -13 & 75 & 14 & 97 & 97 & 97 & 220 & 99 & 346 & 131 & 439 & 161 & 475 & 184 & 475 & 161 & 439 & 131 & 346 & 99 & 220 & 97 & 97 & 97 & 14 & 75 & -13 & 40 & -8 & 13 & -1 & 2 & 0 & 0 \\
-1 & 2 & -13 & 24 & -78 & 129 & -293 & 437 & -836 & 1141 & -1923 & 2472 & -3750 & 4492 & -6241 & 6974 & -8862 & 9247 & -10899 & 10608 & -11644 & 10608 & -10899 & 9247 & -8862 & 6974 & -6241 & 4492 & -3750 & 2472 & -1923 & 1141 & -836 & 437 & -293 & 129 & -78 & 24 & -13 & 2 & -1 \\
3 & -9 & 39 & -92 & 233 & -441 & 872 & -1412 & 2376 & -3452 & 5159 & -6816 & 9305 & -11367 & 14282 & -16253 & 19100 & -20389 & 22510 & -22725 & 23770 & -22725 & 22510 & -20389 & 19100 & -16253 & 14282 & -11367 & 9305 & -6816 & 5159 & -3452 & 2376 & -1412 & 872 & -441 & 233 & -92 & 39 & -9 & 3 \\
-3 & 11 & -42 & 100 & -235 & 437 & -810 & 1277 & -2012 & 2798 & -3938 & 4966 & -6373 & 7424 & -8878 & 9657 & -10881 & 11240 & -12129 & 12016 & -12518 & 12016 & -12129 & 11240 & -10881 & 9657 & -8878 & 7424 & -6373 & 4966 & -3938 & 2798 & -2012 & 1277 & -810 & 437 & -235 & 100 & -42 & 11 & -3 \\
1 & -4 & 15 & -34 & 76 & -126 & 209 & -274 & 354 & -328 & 290 & -40 & -256 & 855 & -1407 & 2278 & -2980 & 3878 & -4357 & 4932 & -4940 & 4932 & -4357 & 3878 & -2980 & 2278 & -1407 & 855 & -256 & -40 & 290 & -328 & 354 & -274 & 209 & -126 & 76 & -34 & 15 & -4 & 1 \\
0 & 0 & -1 & 3 & -8 & 11 & -11 & -13 & 61 & -174 & 354 & -646 & 1001 & -1486 & 2012 & -2596 & 3101 & -3625 & 3991 & -4252 & 4286 & -4252 & 3991 & -3625 & 3101 & -2596 & 2012 & -1486 & 1001 & -646 & 354 & -174 & 61 & -13 & -11 & 11 & -8 & 3 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -4 & 12 & -26 & 48 & -73 & 101 & -118 & 114 & -75 & 12 & 86 & -202 & 317 & -386 & 413 & -386 & 317 & -202 & 86 & 12 & -75 & 114 & -118 & 101 & -73 & 48 & -26 & 12 & -4 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$H_{[21]}^{11a231} = \frac{1}{q^{40}A^{12}} \cdot \tag{175}$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -4 & 2 & -8 & 4 & -13 & 6 & -16 & 8 & -20 & 8 & -16 & 6 & -13 & 4 & -8 & 2 & -4 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 9 & 0 & 19 & -1 & 38 & 1 & 57 & -2 & 77 & 3 & 86 & -4 & 86 & 3 & 77 & -2 & 57 & 1 & 38 & -1 & 19 & 0 & 9 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -3 & -3 & -16 & 6 & -56 & 36 & -151 & 104 & -301 & 218 & -500 & 348 & -686 & 459 & -806 & 496 & -806 & 459 & -686 & 348 & -500 & 218 & -301 & 104 & -151 & 36 & -56 & 6 & -16 & -3 & -3 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & -1 & 13 & -8 & 40 & -15 & 80 & 7 & 109 & 79 & 118 & 195 & 127 & 321 & 154 & 417 & 177 & 462 & 198 & 462 & 177 & 417 & 154 & 321 & 127 & 195 & 118 & 79 & 109 & 7 & 80 & -15 & 40 & -8 & 13 & -1 & 2 & 0 & 0 \\
-1 & 2 & -13 & 24 & -77 & 128 & -294 & 440 & -844 & 1159 & -1951 & 2511 & -3805 & 4557 & -6311 & 7051 & -8938 & 9317 & -10968 & 10673 & -11704 & 10673 & -10968 & 9317 & -8938 & 7051 & -6311 & 4557 & -3805 & 2511 & -1951 & 1159 & -844 & 440 & -294 & 128 & -77 & 24 & -13 & 2 & -1 \\
3 & -9 & 38 & -89 & 227 & -431 & 857 & -1393 & 2354 & -3427 & 5135 & -6793 & 9280 & -11339 & 14248 & -16206 & 19038 & -20312 & 22416 & -22621 & 23664 & -22621 & 22416 & -20312 & 19038 & -16206 & 14248 & -11339 & 9280 & -6793 & 5135 & -3427 & 2354 & -1393 & 857 & -431 & 227 & -89 & 38 & -9 & 3 \\
-3 & 11 & -41 & 97 & -229 & 427 & -795 & 1258 & -1990 & 2773 & -3914 & 4943 & -6348 & 7396 & -8844 & 9610 & -10819 & 11163 & -12035 & 11912 & -12412 & 11912 & -12035 & 11163 & -10819 & 9610 & -8844 & 7396 & -6348 & 4943 & -3914 & 2773 & -1990 & 1258 & -795 & 427 & -229 & 97 & -41 & 11 & -3 \\
1 & -4 & 15 & -34 & 75 & -125 & 210 & -277 & 362 & -346 & 318 & -79 & -201 & 790 & -1337 & 2201 & -2904 & 3808 & -4288 & 4867 & -4880 & 4867 & -4288 & 3808 & -2904 & 2201 & -1337 & 790 & -201 & -79 & 318 & -346 & 362 & -277 & 210 & -125 & 75 & -34 & 15 & -4 & 1 \\
0 & 0 & -1 & 3 & -8 & 11 & -11 & -11 & 56 & -167 & 342 & -628 & 980 & -1461 & 1984 & -2571 & 3078 & -3603 & 3975 & -4239 & 4272 & -4239 & 3975 & -3603 & 3078 & -2571 & 1984 & -1461 & 980 & -628 & 342 & -167 & 56 & -11 & -11 & 11 & -8 & 3 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -4 & 12 & -26 & 48 & -74 & 104 & -123 & 123 & -90 & 33 & 58 & -167 & 277 & -342 & 367 & -342 & 277 & -167 & 58 & 33 & -90 & 123 & -123 & 104 & -74 & 48 & -26 & 12 & -4 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$H_{[21]}^{11n71} = \frac{A^6}{q^{34}} \cdot \tag{176}$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & -1 & 2 & 0 & 2 & -1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -2 & -8 & 0 & -10 & -2 & -19 & -2 & -16 & -2 & -19 & -2 & -10 & 0 & -8 & -2 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3 & 6 & 9 & 16 & 6 & 36 & 13 & 69 & 8 & 85 & 8 & 100 & 8 & 85 & 8 & 69 & 13 & 36 & 6 & 16 & 9 & 6 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -5 & -5 & -25 & 2 & -67 & 20 & -164 & 59 & -287 & 141 & -437 & 209 & -538 & 243 & -538 & 209 & -437 & 141 & -287 & 59 & -164 & 20 & -67 & 2 & -25 & -5 & -5 & -1 & 0 & 0 & 0 \\
0 & 1 & 2 & 10 & 8 & 26 & 23 & 51 & 77 & 78 & 153 & 96 & 239 & 161 & 293 & 218 & 299 & 253 & 299 & 218 & 293 & 161 & 239 & 96 & 153 & 78 & 77 & 51 & 23 & 26 & 8 & 10 & 2 & 1 & 0 \\
0 & -8 & 8 & -46 & 59 & -172 & 205 & -475 & 542 & -1026 & 1123 & -1866 & 1926 & -2863 & 2796 & -3722 & 3363 & -4080 & 3363 & -3722 & 2796 & -2863 & 1926 & -1866 & 1123 & -1026 & 542 & -475 & 205 & -172 & 59 & -46 & 8 & -8 & 0 \\
1 & 10 & -22 & 79 & -142 & 334 & -506 & 921 & -1257 & 1966 & -2431 & 3408 & -3892 & 4954 & -5259 & 6203 & -6111 & 6655 & -6111 & 6203 & -5259 & 4954 & -3892 & 3408 & -2431 & 1966 & -1257 & 921 & -506 & 334 & -142 & 79 & -22 & 10 & 1 \\
-1 & -3 & 11 & -46 & 89 & -204 & 333 & -573 & 773 & -1134 & 1368 & -1773 & 1952 & -2357 & 2418 & -2747 & 2675 & -2894 & 2675 & -2747 & 2418 & -2357 & 1952 & -1773 & 1368 & -1134 & 773 & -573 & 333 & -204 & 89 & -46 & 11 & -3 & -1 \\
0 & 0 & 1 & 4 & -9 & 22 & -30 & 54 & -38 & 4 & 108 & -237 & 479 & -700 & 987 & -1173 & 1362 & -1371 & 1362 & -1173 & 987 & -700 & 479 & -237 & 108 & 4 & -38 & 54 & -30 & 22 & -9 & 4 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -3 & 14 & -37 & 79 & -161 & 278 & -431 & 605 & -792 & 947 & -1058 & 1093 & -1058 & 947 & -792 & 605 & -431 & 278 & -161 & 79 & -37 & 14 & -3 & -1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$H_{[21]}^{11n75} = \frac{A^6}{q^{34}} \cdot \tag{177}$$

$$\left(\begin{smallmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & -1 & 2 & 0 & 2 & -1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -2 & -8 & 0 & -10 & -2 & -19 & -2 & -16 & -2 & -19 & -2 & -10 & 0 & -8 & -2 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3 & 6 & 9 & 16 & 5 & 38 & 11 & 73 & 2 & 91 & 1 & 108 & 1 & 91 & 2 & 73 & 11 & 38 & 5 & 16 & 9 & 6 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -5 & -5 & -25 & 4 & -70 & 22 & -169 & 65 & -290 & 145 & -442 & 209 & -538 & 247 & -538 & 209 & -442 & 145 & -290 & 65 & -169 & 22 & -70 & 4 & -25 & -5 & -5 & -1 & 0 & 0 & 0 \\
0 & 1 & 2 & 10 & 7 & 26 & 25 & 49 & 82 & 68 & 163 & 86 & 255 & 149 & 300 & 206 & 308 & 249 & 308 & 206 & 300 & 149 & 255 & 86 & 163 & 68 & 82 & 49 & 25 & 26 & 7 & 10 & 2 & 1 & 0 \\
0 & -8 & 9 & -48 & 62 & -176 & 210 & -479 & 545 & -1030 & 1124 & -1868 & 1932 & -2871 & 2806 & -3738 & 3382 & -4096 & 3382 & -3738 & 2806 & -2871 & 1932 & -1868 & 1124 & -1030 & 545 & -479 & 210 & -176 & 62 & -48 & 9 & -8 & 0 \\
1 & 10 & -23 & 81 & -145 & 338 & -511 & 925 & -1260 & 1970 & -2432 & 3410 & -3898 & 4962 & -5269 & 6219 & -6130 & 6671 & -6130 & 6219 & -5269 & 4962 & -3898 & 3410 & -2432 & 1970 & -1260 & 925 & -511 & 338 & -145 & 81 & -23 & 10 & 1 \\
-1 & -3 & 11 & -46 & 90 & -204 & 331 & -571 & 768 & -1124 & 1358 & -1763 & 1936 & -2345 & 2411 & -2735 & 2666 & -2890 & 2666 & -2735 & 2411 & -2345 & 1936 & -1763 & 1358 & -1124 & 768 & -571 & 331 & -204 & 90 & -46 & 11 & -3 & -1 \\
0 & 0 & 1 & 4 & -9 & 22 & -30 & 52 & -35 & 2 & 113 & -243 & 482 & -704 & 992 & -1173 & 1362 & -1375 & 1362 & -1173 & 992 & -704 & 482 & -243 & 113 & 2 & -35 & 52 & -30 & 22 & -9 & 4 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -3 & 14 & -37 & 79 & -160 & 276 & -429 & 601 & -786 & 941 & -1051 & 1085 & -1051 & 941 & -786 & 601 & -429 & 276 & -160 & 79 & -37 & 14 & -3 & -1 & 0 & 0 & 0 & 0 & 0
\end{smallmatrix}\right)$$